THE OBSTACLE PROBLEM FOR A FRACTIONAL MONGE–AMPÈRE EQUATION

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Abstract. We study the obstacle problem for a nonlocal, degenerate elliptic Monge–Ampère equation. We show existence and regularity of a unique classical solution to the problem and regularity of the free boundary.

1. Introduction

Obstacle problems for nonlocal operators appear in optimal control, mathematical finance, biology, and elasticity, among other applied sciences. The regularity of solutions and free boundaries for this type of nonlinear problem for the fractional Laplacian was studied by Silvestre in [14], and by Caffarelli–Salsa–Silvestre in [3], and for homogeneous, translation invariant, purely nonlocal uniformly elliptic operators by Caffarelli–Ros-Oton–Serra in [2].

In this paper, we investigate the following nonlocal obstacle problem:

$$\begin{cases} \mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\ \lim\limits_{|x|\to\infty} (u-\phi)(x) = 0. \end{cases} \tag{1.1}$$

Here the fractional Monge–Ampère operator $\mathcal{D}_s$ is defined for $s \in (1/2, 1)$ and $u : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, as

$$\mathcal{D}_s u(x) = \inf_{A \in \mathcal{M}} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|A^{-1}y|^{n+2s}}\, dy \tag{1.2}$$

where $x \in \mathbb{R}^n$, $c_{n,s} > 0$, and $\mathcal{M}$ is the class of all positive definite symmetric matrices A of size $n \times n$ such that $\det A = 1$. In addition, ϕ is a function in $C^{2,\sigma}(\mathbb{R}^n)$, $\sigma > 0$, that is strictly convex in compact sets and asymptotically linear at infinity, and conditions on the obstacle ψ are given below.

The nonlocal operator (1.2) was first introduced by L. A. Caffarelli and F. Charro in [1] as a fractional analogue to the classical Monge–Ampère operator. In fact, if u is a convex C^2 function, then it can be checked that

$$n(\det D^2 u(x))^{1/n} = \inf_{A \in \mathcal{M}} \operatorname{tr}(A^2 D^2 u)(x). \tag{1.3}$$

If, in addition, u is asymptotically linear at infinity, then

$$\lim_{s \to 1} \mathcal{D}_s u(x) = n(\det D^2 u(x))^{1/n}$$

(see [1]). Like its local counterpart (1.3), the fractional operator (1.2) is degenerate elliptic. Indeed, matrices of the form $A = \operatorname{diag}(\varepsilon, 1/\varepsilon)$, $\varepsilon > 0$, in dimension 2, are in $\mathcal{M}$, and they degenerate as $\varepsilon \searrow 0$. Thus, the existence and regularity theory for nonlocal elliptic equations previously developed

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in [4, 5, 6], see also [9, 11, 13], does not directly apply to equations involving (1.2). Caffarelli and Charro considered the problem

$$\begin{cases} \mathcal{D}_s \bar{u}(x) = \bar{u}(x) - \phi(x) & \text{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} (\bar{u} - \phi)(x) = 0 \end{cases} \tag{1.4}$$

where ϕ is as above, and they showed existence of a unique, globally Lipschitz and semiconcave classical solution $\bar{u}$. In addition, it was proved in [1] that $\bar{u}$ has the crucial property that $\bar{u} > \phi$ in $\mathbb{R}^n$. This ultimately implies that $\mathcal{D}_s$, when acting on u, is locally uniformly elliptic and, consequently, $\nabla\bar{u}$ is locally Hölder continuous.

In view of the comparison principle for (1.4) and in order to have a meaningful obstacle problem, we assume that the obstacle $\psi \in C^{2,1}(\mathbb{R}^n)$ is such that

$$\psi > \phi \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \psi \le \bar{u} \quad \text{in some compact set } \mathcal{K}.$$

Here and in the remainder of this work, $\bar{u}$ denotes the solution to (1.4).

While the main tool in [8] is the solution of a purely nonlocal and degenerate elliptic obstacle problem, which stands as a replacement for the convex envelope in the fractional setting (with the usual convex envelope being such an obstacle solution in the second order setting), we stress that the problem (1.1) is different in nature than the one in [8]. Obstacle problems for the local Monge–Ampère equation (1.3) were considered by Lee [10] and Savin [12]. Our problem (1.1) can be seen as a parallel to [10].

Our first result establishes the existence and global regularity of a unique classical solution to (1.1). For the necessary notation, see section 2.

Theorem 1.1. *There exists a unique classical solution u to the obstacle problem* (1.1)*. Moreover, u is globally Lipschitz continuous and semiconcave with constants no larger than*

$$M_1 = \max\left\{[\phi]_{\mathrm{Lip}(\mathbb{R}^n)}, [\psi]_{\mathrm{Lip}(\mathbb{R}^n)}\right\} \quad \text{and} \quad M_2 = \max\left\{\mathrm{SC}(\phi), \mathrm{SC}(\psi)\right\}, \tag{1.5}$$

respectively, and the contact set $\{u = \psi\} \subset \mathcal{K}$ is compact. Furthermore,

$$u > \phi \quad \text{in } \mathbb{R}^n. \tag{1.6}$$

The degenerate ellipticity of the fractional Monge–Ampère operator (1.2) prevents us from applying standard techniques used to prove existence and uniqueness for nonlocal uniformly elliptic obstacle problems [2, 14]. Therefore, to construct the solution u to (1.1), we need to devise a new strategy. This is one of the main contributions of this paper. To prove Theorem 1.1, we consider a family of obstacle problems of the form (1.1), but where $\mathcal{D}_s$ is replaced by a truncated operator $\mathcal{D}_s^\varepsilon$ (see section 2). We build the solutions u_ε to such uniformly elliptic nonlocal problems as the largest subsolution sitting below ψ (see Theorem 3.8), following the standard method. More importantly, the key feature of the family of solutions $\{u_\varepsilon\}_{\varepsilon>0}$ is that it is uniformly globally Lipschitz continuous and semiconcave with constants no larger than M_1 and M_2 (see (1.5)), respectively. At this point, though, the degenerate ellipticity of $\mathcal{D}_s$ prevents us from applying the stability of viscosity solutions under local uniform convergence. Our crucial, delicate idea, which allows us to overcome this difficulty, is to show that $u = \inf_{\varepsilon>0} u_\varepsilon$ remains strictly above ϕ (see Lemma 3.9). See section 3 for details.

Next, we prove local Hölder estimates on ∇u outside of the contact set $\{u = \psi\}$ and across the free boundary $\partial\{u < \psi\}$.

Theorem 1.2. *Let u be the solution to the obstacle problem* (1.1)*.*

(1) Let $\mathcal{O}$ be an open set and let $\mathcal{O}_\delta$, $\delta > 0$, be a δ-neighborhood of $\mathcal{O}$ such that $\mathcal{O}_\delta \subset\subset \{u < \psi\}$. There exists $\beta = \beta(n, s, \inf_{\mathcal{O}_\delta}(u - \phi), M_1, M_2) \in (0, 1)$ such that $u \in C^{1,2s+\beta-1}(\mathcal{O})$ and

$$\|u\|_{C^{1,2s+\beta-1}(\mathcal{O})} \le C(1 + \|u - \phi\|_{L^\infty(\mathbb{R}^n)}),$$

where $C = C(\beta, \mathrm{diam}(\mathcal{O})) > 0$.

(2) *Let* $\mathcal{B}$ *be a ball centered at the origin such that* $\{u = \psi\} \subset \mathcal{B}$. *There exists* $\tau = \tau(n, s, \inf_{4\mathcal{B}}(u - \phi), M_1, M_2) \in (0,1)$ *such that* $u \in C^{1,\tau}(\mathcal{B})$ *and*

$$\|u\|_{C^{1,\tau}(\mathcal{B})} \leq C(1 + \|\psi\|_{C^{1,\tau}(\mathcal{B})}),$$

where $C = C(\tau, \mathrm{diam}(\mathcal{B})) > 0$.

The separation property (1.6) and the global regularity of u we found in Theorem 1.1 will permit us to prove that, if we fix any ball $\mathcal{B}$, then u solves

$$\begin{cases} \mathcal{D}_s^\lambda u \geq u - \phi & \text{in } \mathbb{R}^n \\ u \leq \psi & \text{in } \mathbb{R}^n \\ \mathcal{D}_s^\lambda u = u - \phi & \text{in } \{u < \psi\} \cap \mathcal{B} \\ \lim\limits_{|x| \to \infty} (u - \phi)(x) = 0 \end{cases} \tag{1.7}$$

where $\mathcal{D}_s^\lambda$ is the truncated version of (1.2), with ellipticity constants depending on the gap between u and ϕ in $\mathcal{B}$. Then Theorem 1.2 will follow from local regularity estimates for uniformly elliptic nonlocal equations. These ideas also demonstrate another important point of divergence between our obstacle problem and uniformly elliptic nonlocal obstacle problems. In [2], solutions are shown to be $C^{1,\tau}(\mathbb{R}^n)$. In contrast, since $\lim_{|x|\to\infty}(u - \phi)(x) = 0$, we cannot guarantee that $\mathcal{D}_s$, when acting on u, will be globally uniformly elliptic. In particular, the Hölder exponents β and τ in Theorem 1.2 degenerate as $\mathcal{O}$ drifts to infinity and $\mathcal{B}$ increases in size, respectively.

To study the regularity of the free boundary $\partial\{u < \psi\}$ and the behavior of u near free boundary points, we fix a ball $\mathcal{B}$ centered at the origin such that $\{u = \psi\} \subset \mathcal{B}$. Then, u satisfies the obstacle problem (1.7). Let

$$v = \psi - u. \tag{1.8}$$

Let $x_0 \in \partial\{u < \psi\} = \partial\{v > 0\}$ be a regular free boundary point (see Definition 5.1). As in [2], for $r > 0$ and $\alpha \in (0,1)$ sufficiently small, we define the rescalings

$$v_r(x) = \frac{v(x_0 + rx)}{r^{1+s+\alpha}\theta(x_0, r)} \quad \text{for } x \in \mathbb{R}^n$$

where

$$\theta(x_0, r) = \sup_{\rho \geq r} \frac{\|\nabla v(x_0 + \cdot)\|_{L^\infty(B_\rho)}}{\rho^{s+\alpha}}.$$

Theorem 1.3. *There exist a sequence* $r_k \searrow 0$, $1/4 \leq K_0 \leq 1$, *and* $\mathrm{e}_0 \in \mathbb{S}^{n-1}$ *such that*

$$v_{r_k}(x) \to K_0(\mathrm{e}_0 \cdot x)_+^{1+s} \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n), \text{ as } k \to \infty.$$

Observe that, unlike in [2], we do not know whether or not $\mathcal{D}_s^\lambda u = u - \phi$ in the part of the noncoincidence set $\{u < \psi\}$ that lies outside of $\mathcal{B}$. Hence, even after subtracting the obstacle from u in (1.7), we cannot use known regularity results for the free boundary. In fact, it is well known that the behavior at infinity of solutions to nonlocal equations can have dramatic consequences on their local properties (see, for instance, [7]). Moreover, our Hölder estimates for ∇u degenerate at infinity. Nonetheless, we are able to overcome these issues due to the global regularity of u we proved in Theorem 1.1. Indeed, this gives us enough control at infinity to be able to show Theorem 1.3. Finally, the separation property (1.6), the global regularity of u, and Theorem 1.3 permit us to use the methods of [2], in our setting, to obtain regularity of the free boundary.

Theorem 1.4. *Let* u *be the solution to* (1.1). *Let* $\mathcal{B}$ *be a ball centered at the origin such that* $\{u = \psi\} \subset \mathcal{B}$. *There exists* $\bar{\alpha} = \bar{\alpha}(n, s, \inf_{4\mathcal{B}}(u - \phi), M_1, M_2) \in (0,1)$ *such that the following holds: for any* $\gamma \in (0, \bar{\alpha})$ *and* $\alpha \in (0, \bar{\alpha})$ *such that* $1 + s + \alpha < 2$ *and for any* $x_0 \in \partial\{u < \psi\}$, *(1) either*

$$\liminf_{r\searrow 0} \frac{|\{u=\psi\}\cap B_r(x_0)|}{|B_r(x_0)|} > 0 \quad \textit{and}$$

$$\psi(x)-u(x) = cd^{1+s}(x) + o\big(|x-x_0|^{1+s+\alpha}\big)$$

(2) or

$$\liminf_{r\searrow 0} \frac{|\{u=\psi\}\cap B_r(x_0)|}{|B_r(x_0)|} = 0 \quad \textit{and}$$

$$\psi(x)-u(x) = o\big(|x-x_0|^{\min\{2s+\gamma,1+s+\alpha\}}\big)$$

(3) or

$$\liminf_{r\searrow 0} \frac{|\{u=\psi\}\cap B_r(x_0)|}{|B_r(x_0)|} > 0 \quad \textit{and}$$

$$\psi(x)-u(x) = o\big(|x-x_0|^{1+s+\alpha}\big)$$

where $d(x)=\operatorname{dist}(x,\partial\{u<\psi\})$ *and* $c>0$. *Moreover, the set of points* x_0 *satisfying (1) is an open subset of the free boundary of class* $C^{1,\gamma}$.

The paper is organized as follows. In section 2, we establish some preliminary results that will be needed for the rest of the work. The proofs of Theorems 1.1, 1.2, 1.3, and 1.4 are presented in sections 3, 4, 5, and 6, respectively.

2. Preliminaries

In this section, we recall some facts about the fractional Monge–Ampère operator $\mathcal{D}_s$, problem (1.4), and uniformly elliptic nonlocal operators.

2.1. **Notation.** Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and $f:\mathcal{O}\to\mathbb{R}$. We denote the Lipschitz constant of f in $\mathcal{O}$ by

$$[f]_{\mathrm{Lip}(\mathcal{O})} = \sup_{x,y\in\mathcal{O},\, x\neq y} \frac{|f(x)-f(y)|}{|x-y|}.$$

For the second order incremental quotient of f at x in the direction of y, we write

$$\delta(f,x,y) = f(x+y)+f(x-y)-2f(x).$$

When $\mathcal{O}=\mathbb{R}^n$, we say that f is semiconcave if there exists a constant $C>0$ such that $\delta(f,x,y)\leq C|y|^2$ for all $x,y\in\mathbb{R}^n$. In this case,

$$\mathrm{SC}(f) = \sup_{x,y\in\mathbb{R}^n} \frac{\delta(f,x,y)}{|y|^2}$$

is the semiconcavity constant of f. Alternatively, f is semiconcave if and only if $f(x)-C|x|^2/2$ is concave.

Let $\mathrm{USC}(\mathcal{O})$ (resp. $\mathrm{LSC}(\mathcal{O})$) be the set of functions that are upper (resp. lower) semicontinuous in $\mathcal{O}$. Define

$$f^*(x) = \lim_{r\to 0}\sup\big\{f(y) : y\in\mathcal{O} \text{ and } |y-x|<r\big\} \quad \text{for every } x\in\mathcal{O}.$$

We call f^* the upper semicontinuous envelope of f in $\mathcal{O}$; it is the smallest $g\in\mathrm{USC}(\mathcal{O})$ satisfying $f\leq g$.

Remark 2.1. A simple, useful property of the upper semicontinuous envelope f^* of f in $\mathcal{O}$ is that for any $x_0\in\mathcal{O}$, there exist points $y_k\in\mathcal{O}$ such that $y_k\to x_0$ and $f(y_k)\to f^*(x_0)$, as $k\to\infty$. (Note, we allow $y_k=x_0$ for all k.)

2.2. **The fractional Monge–Ampère operator.** We begin this subsection by providing some novel insight on the definition of the fractional Monge–Ampère operator $\mathcal{D}_s u$ in (1.2), which may be of independent interest. Next, we precisely describe ϕ. Then, we discuss the definition of viscosity solutions and some further properties of $\mathcal{D}_s u$ and the problem (1.4).

Recall that

$$\mathcal{M} = \big\{\text{symmetric positive definite matrices } A \text{ of size } n \times n \text{ such that } \det A = 1\big\}.$$

For any $A \in \mathcal{M}$, we define the constant coefficient second order elliptic operator

$$L_A w(x) = -\Delta[w \circ A](A^{-1}x) = -\operatorname{tr}(A^2 D^2 w)(x),$$

see (1.3). Then, L_A is nothing but a linear transformation of the Laplacian $-\Delta$. For $s \in (0,1)$, consider the fractional power operator

$$L_A^s = -(L_A)^s \quad \text{in } \mathbb{R}^n.$$

Lemma 2.2. *Let $w : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{(1+|x|)^{n+2s}}\, dx < \infty.$$

Let $\mathcal{O}$ be an open set. If $w \in C^{2s+\delta}(\mathcal{O})$ or $w \in C^{1,2s+\delta-1}(\mathcal{O})$ when $s \geq 1/2$, for some $\delta > 0$, then, for any $x \in \mathcal{O}$,

$$\begin{aligned} L_A^s w(x) &= c_{n,s}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{w(y)-w(x)}{|A^{-1}(y-x)|^{n+2s}}\, dy \\ &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{w(x+y)+w(x-y)-2w(x)}{|A^{-1}y|^{n+2s}}\, dy \\ &= -(-\Delta)^s[w \circ A](A^{-1}x), \end{aligned} \tag{2.1}$$

where $c_{n,s} = \frac{4^s\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|} > 0$. As a consequence of (2.1), we have

$$\mathcal{D}_s w(x) = \inf\big\{L_A^s w(x) : A \in \mathcal{M}\big\}.$$

Proof. The idea is to first prove (2.1) for w in the Schwartz class $\mathcal{S}$, by applying the method of semigroups as in [15, Lemma 5.1]. Then, for w as in the hypotheses, one can use an approximation device exactly as done in [14, Proposition 2.4]. We just sketch the steps here. For $A \in \mathcal{M}$ and $w \in \mathcal{S}$, the heat semigroup generated by L_A acting on w is given explicitly by

$$e^{tL_A} w(x) = \int_{\mathbb{R}^n} \frac{e^{-|A^{-1}x|^2/(4t)}}{(4\pi t)^{n/2}}\, w(x-y)\, dy,$$

for $x \in \mathbb{R}^n$ and $t > 0$. Then, since $e^{tL_A}1(x) = 1$, by Fubini's theorem (see [15, Lemma 5.1]) and the change of variables $r = |A^{-1}x|^2/(4t)$,

$$\begin{aligned} L_A^s w(x) &= -(L_A)^s w(x) \\ &= \frac{1}{|\Gamma(-s)|} \int_0^\infty \big(e^{tL_A} w(x) - w(x)\big)\, \frac{dt}{t^{1+s}} \\ &= \frac{1}{|\Gamma(-s)|} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-|A^{-1}x|^2/(4t)}}{(4\pi t)^{n/2}} (w(x-y)-w(x))\, dy\, \frac{dt}{t^{1+s}} \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} (w(x-y)-w(x)) \bigg[\int_0^\infty \frac{e^{-|A^{-1}x|^2/(4t)}}{|\Gamma(-s)|(4\pi t)^{n/2}}\, \frac{dt}{t^{1+s}}\bigg]\, dy \\ &= c_{n,s}\, \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{w(x-y)-w(x)}{|A^{-1}y|^{n+2s}}\, dy, \end{aligned}$$

as desired. The second identity in (2.1) follows immediately from the first one, and the third one is deduced via a simple change of variables. □

Now, we give the precise description of the function ϕ appearing in (1.4) and (1.1). Let Γ be a cone and $\eta : \mathbb{R}^n \to \mathbb{R}$ be such that

$$|\eta(x)| \leq a|x|^{-\epsilon}, \quad |\nabla\eta(x)| \leq a|x|^{-(1+\epsilon)}, \quad \text{and} \quad |D^2\eta(x)| \leq a|x|^{-(2+\epsilon)}$$

for some constants $a > 0$ and $\epsilon \in (0, n)$. We let $\phi \in C^{2,\sigma}(\mathbb{R}^n)$, for some $\sigma > 0$, be such that

$$\phi(0) = 0, \quad \nabla\phi(0) = 0, \quad \text{and} \quad \phi = \Gamma + \eta \quad \text{near infinity.}$$

We will work with viscosity solutions as defined in [1, Definition 2.1].

Definition 2.3. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$. A function $w : \mathbb{R}^n \to \mathbb{R}$ such that $w \in \mathrm{USC}(\overline{\mathcal{O}})$ (resp. $w \in \mathrm{LSC}(\overline{\mathcal{O}})$) is called a viscosity subsolution (resp. supersolution) to $\mathcal{D}_s w = w - \phi$ in $\mathcal{O}$, which we denote by

$$\mathcal{D}_s w \geq w - \phi \quad \big(\text{resp. } \mathcal{D}_s w \leq w - \phi\big) \quad \text{in } \mathcal{O},$$

if whenever

- x_0 is a point in $\mathcal{O}$;
- $\mathcal{N} \subset \mathcal{O}$ is an open neighborhood of x_0;
- P is a C^2 function on $\overline{\mathcal{N}}$;
- $P(x_0) = w(x_0)$; and
- $P(x) > w(x)$ (resp. $P(x) < w(x)$) for every $x \in \mathcal{N} \setminus \{x_0\}$;

then

$$\mathcal{D}_s\vartheta(x_0) \geq \vartheta(x_0) - \phi(x_0) \quad (\text{resp. } \mathcal{D}_s\vartheta(x_0) \leq \vartheta(x_0) - \phi(x_0))$$

where ϑ is defined as

$$\vartheta(x) = \begin{cases} P(x) & \text{if } x \in \mathcal{N} \\ w(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}. \end{cases} \tag{2.2}$$

When all of the items listed above are satisfied for some triplet $(P, x_0, \mathcal{N})$, we say that P is a C^2 function touching w from above (resp. below) at x_0 in $\mathcal{N}$. A viscosity solution w is both a viscosity subsolution and a viscosity supersolution. In particular, solutions are continuous by definition.

From now on, any reference to a subsolution, supersolution, or solution will be in the viscosity sense.

Note that a semiconcave function can always be touched from above by a quadratic polynomial at any point.

Remark 2.4. Let P be a C^2 function touching w from above (resp. below) at x_0 in $\mathcal{N}$. If $\mathcal{N}'$ is any open subset of $\mathcal{N}$ containing x_0, then P is a C^2 function that touches w from above (resp. below) at x_0 in $\mathcal{N}'$. Define ϑ as in (2.2) and let

$$\vartheta'(x) = \begin{cases} P(x) & \text{if } x \in \mathcal{N}' \\ w(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}'. \end{cases}$$

Then, $\vartheta \geq \vartheta'$ (resp. $\vartheta \leq \vartheta'$) in $\mathbb{R}^n$ and $\vartheta(x_0) = \vartheta'(x_0)$, so that $\delta(\vartheta, x_0, y) \geq \delta(\vartheta', x_0, y)$ for every $y \in \mathbb{R}^n$ (resp. $\delta(\vartheta, x_0, y) \leq \delta(\vartheta', x_0, y)$). It follows that $L_A^s\vartheta(x_0) \geq L_A^s\vartheta'(x_0)$ (resp. $L_A^s\vartheta(x_0) \leq L_A^s\vartheta'(x_0)$) for all matrices $A \in \mathcal{M}$, which implies that

$$\mathcal{D}_s\vartheta(x_0) \geq \mathcal{D}_s\vartheta'(x_0)$$

(resp. $\mathcal{D}_s\vartheta(x_0) \leq \mathcal{D}_s\vartheta'(x_0)$), see (1.2). Therefore, if $\mathcal{D}_s w \geq w - \phi$ (resp. $\mathcal{D}_s w \leq w - \phi$) in $\mathbb{R}^n$, then, in order to check the viscosity solution condition from Definition 2.3, we can always restrict ourselves to working in a smaller neighborhood $\mathcal{N}' \subset \mathcal{N}$ containing x_0.

From the definition of $\mathcal{D}_s$, we see that

(1) If $\tau_h w(x) = w(x+h)$, for some $h \in \mathbb{R}^n$, then $\mathcal{D}_s(\tau_h w) = \tau_h(\mathcal{D}_s w)$.
(2) For any constant $c \in \mathbb{R}$, $\mathcal{D}_s(w+c) = \mathcal{D}_s w$.
(3) $\mathcal{D}_s$ is a concave operator in the sense that, for any w_1, w_2,

$$\mathcal{D}_s\left(\frac{w_1+w_2}{2}\right) \geq \frac{1}{2}\mathcal{D}_s w_1 + \frac{1}{2}\mathcal{D}_s w_2.$$

Let w be a viscosity subsolution (resp. supersolution) as in Definition 2.3. In the next lemma, we state that if w can be touched from above (resp. below) by a C^2 function at a point x, then $\mathcal{D}_s w(x)$ can be computed classically. This is an important, typical feature of nonlocal equations, see also [5, Lemma 3.3].

Lemma 2.5 (see [1, Lemma 2.2]). *Let $w : \mathbb{R}^n \to \mathbb{R}$ be asymptotically linear at infinity. If*

$$\mathcal{D}_s w \geq w - \phi \quad \big(\textit{resp. } \mathcal{D}_s w \leq w - \phi\big) \quad \textit{in } \mathcal{O} \subset \mathbb{R}^n$$

in the viscosity sense and w can be touched by a C^2 function from above (resp. below) at a point $x \in \mathcal{O}$, then $\delta(w,x,y)/|A^{-1}y|^{n+2s} \in L^1(\mathbb{R}^n)$ for every $A \in \mathcal{M}$ and

$$\mathcal{D}_s w(x) \geq w(x) - \phi(x) \quad \big(\textit{resp. } \mathcal{D}_s w(x) \leq w(x) - \phi(x)\big)$$

in the classical sense.

Finally, we recall the comparison principle proved by Caffarelli and Charro.

Theorem 2.6 (see [1, Theorem 4.1]). *Let $w_1 \in \mathrm{USC}(\mathbb{R}^n)$ and $w_2 \in \mathrm{LSC}(\mathbb{R}^n)$ such that*

$$\begin{cases} \mathcal{D}_s w_1 \geq w_1 - \phi & \textit{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} (w_1 - \phi)(x) = 0 \end{cases} \quad \textit{and} \quad \begin{cases} \mathcal{D}_s w_2 \leq w_2 - \phi & \textit{in } \mathbb{R}^n \\ \lim\limits_{|x|\to\infty} (w_2 - \phi)(x) = 0. \end{cases}$$

Then,

$$w_1 \leq w_2 \quad \textit{in } \mathbb{R}^n.$$

2.3. **The truncated fractional Monge–Ampère operator.** For $\varepsilon > 0$, consider the class

$$\mathcal{M}_\varepsilon = \big\{A \in \mathcal{M} : \langle A\xi, \xi\rangle \geq \varepsilon|\xi|^2 \text{ for all } \xi \in \mathbb{R}^n\big\}.$$

Since the matrices in $\mathcal{M}$ have determinant one, not only are the eigenvalues of $A \in \mathcal{M}_\varepsilon$ bounded from below, but they are also bounded from above. In particular,

$$\varepsilon|\xi|^2 \leq \langle A\xi, \xi\rangle \leq \varepsilon^{1-n}|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n.$$

We define

$$\mathcal{D}_s^\varepsilon u(x) = \inf_{A \in \mathcal{M}_\varepsilon} L_A^s u(x).$$

The kernels of L_A^s, for $A \in \mathcal{M}_\varepsilon$, satisfy

$$\frac{\varepsilon^{n+2s}}{|y|^{n+2s}} \leq \frac{1}{|A^{-1}y|^{n+2s}} \leq \frac{\varepsilon^{(1-n)(n+2s)}}{|y|^{n+2s}}. \tag{2.3}$$

Therefore, the truncated operator $\mathcal{D}_s^\varepsilon$ is uniformly elliptic in the sense of Caffarelli–Silvestre (see Lemma 2.8 and [5, Definition 3.1, Lemma 3.2]). We define the notion of viscosity subsolution, supersolution, and solution for $\mathcal{D}_s^\varepsilon$ exactly as in Definition 2.3. Moreover, by [5, Lemma 3.3], Lemma 2.5 also holds for $\mathcal{D}_s^\varepsilon$ in place of $\mathcal{D}_s$. Obviously, $\mathcal{D}_s^0 = \mathcal{D}_s$.

Caffarelli and Charro proved in [1, Theorem 3.1] that the operator $\mathcal{D}_s$ becomes uniformly elliptic provided that $\mathcal{D}_s w$ is bounded below away from zero and w is globally Lipschitz and semiconcave. The next statement presents an important refinement that will be crucial to proving our results.

Theorem 2.7. *Let η_0, L, and C be positive constants, and fix an open set $\mathcal{O} \subset \mathbb{R}^n$. There exists $\lambda = \lambda(n, s, \eta_0, L, C) > 0$ such that for any Lipschitz and semiconcave function w with constants L and C, respectively, if $0 \leq \varepsilon < \lambda$ and*

$$\mathcal{D}_s^\varepsilon w \geq \eta_0 > 0 \quad \text{in } \mathcal{O}$$

in the viscosity sense, then

$$\mathcal{D}_s^\varepsilon w(x) = \mathcal{D}_s^\lambda w(x) \quad \text{for every } x \in \mathcal{O}$$

in the classical sense.

Proof. The proof follows by precisely tracking the constants in the proof of Theorem 3.1 in [1]. Following their notation, fix

$$\epsilon = \left(\frac{\eta_0}{2C_1}\right)^{\frac{n-1}{2s}} \quad \text{and} \quad 0 < \theta < \left(\frac{\mu_0}{n\mu_1}\right)^{\frac{n-1}{2s}}$$

(this value of ϵ is not to be confused with $0 \leq \varepsilon < \lambda$ in our hypotheses). Then, ϵ and θ depend only on n, s, η_0, L, and C. Choose $\lambda = \min\{\epsilon, \theta, 1\}$. We notice that if $0 \leq \varepsilon < \lambda$, then we can apply [1, Lemma 3.9] to deduce Proposition 3.3 in [1] with $\mathcal{D}_s^\varepsilon$ in place of $\mathcal{D}_s$. Thus, the statement of [1, Proposition 3.5], being a simple consequence of Proposition 3.3, holds in our setting. We end our proof exactly as in the proof of Theorem 3.1 in [1, pp. 12-13], in which $\theta = 1/k$. ☐

We close this section with some continuity and stability results, whose proofs follow as in [14, Propositions 2.4 and 2.6] and [5, Lemma 4.5] by using that w satisfies (2.4) instead of being just bounded.

Lemma 2.8. *Let $\mathcal{O}$ be an open set, $1/2 < s < 1$, and $w \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that*

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{(1+|x|)^{n+2s}}\,dx < \infty. \tag{2.4}$$

Suppose that $w \in C^{1,2s+\mu-1}(\mathcal{O})$ for some $\mu > 0$. Then, for any positive definite symmetric matrix A of size $n \times n$, $L_A^s w \in C^\mu(\mathcal{O})$ and

$$[L_A^s w]_{C^\mu(\mathcal{O})} \leq C[\nabla w]_{C^{2s+\mu-1}(\mathcal{O})},$$

where $C > 0$ depends only on n, s, μ and the largest eigenvalue of A. In particular, if $\varepsilon > 0$, then

$$\text{the family } \{L_A^s w : A \in \mathcal{M}_\varepsilon\} \text{ is equicontinuous in } \mathcal{O}. \tag{2.5}$$

Consequently, by taking the infimum over $A \in \mathcal{M}_\varepsilon$ above,

$$\mathcal{D}_s^\varepsilon w \in C(\mathcal{O}).$$

We say that a sequence $w_k \in \mathrm{LSC}(\mathbb{R}^n)$, $k \geq 1$, Γ-converges to w in a set $\mathcal{O}$ if the following two conditions hold:

- For any sequence $x_k \to x$ in $\mathcal{O}$, $\liminf_{k\to\infty} w_k(x_k) \geq w(x)$.
- For any $x \in \mathcal{O}$, there is a sequence $x_k \to x$ in $\mathcal{O}$ so that $\limsup_{k\to\infty} w_k(x_k) = w(x)$.

Lemma 2.9. *Let $w_k \in \mathrm{LSC}(\mathbb{R}^n) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a sequence of functions such that*

$$\int_{\mathbb{R}^n} \frac{|w_k(x)|}{(1+|x|)^{n+2s}}\,dx \leq C < \infty$$

for all $k \geq 1$. Let I be either L_A^s, for $A \in \mathcal{M}$, or $\mathcal{D}_s^\varepsilon$, for $\varepsilon > 0$, for any $1/2 < s < 1$. Suppose that

- *$Iw_k \leq f_k$ in $\mathcal{O}$;*
- *$w_k \to w$ in the Γ sense in $\mathcal{O}$;*
- *$w_k \to w$ a.e. in $\mathbb{R}^n$; and*
- *$f_k \to f$ locally uniformly in $\mathcal{O}$ for some continuous function f.*

Then,

$$Iw \leq f \quad \textit{in } \mathcal{O}.$$

An analogous statement to Lemma 2.9 holds for subsolutions.

3. Proof of Theorem 1.1

To construct the solution u, we define the class

$$\mathcal{F} = \Big\{ w \in \mathrm{USC}(\mathbb{R}^n) : \mathcal{D}_s w \geq w - \phi \text{ in } \mathbb{R}^n,\ w \leq \psi, \text{ and } \lim_{|x|\to\infty} (w-\phi)(x) \leq 0 \Big\}. \tag{3.1}$$

Notice that $\mathcal{F}$ is nonempty because $\phi \in \mathcal{F}$. Indeed, by assumption, $\phi < \psi$ in $\mathbb{R}^n$, and by convexity, $\delta(\phi, x, y) \geq 0$ for every $x, y \in \mathbb{R}^n$. Hence, $L^s_A \phi \geq 0$ in $\mathbb{R}^n$ for every $A \in \mathcal{M}$, which implies that $\mathcal{D}_s \phi \geq 0 = \phi - \phi$ on $\mathbb{R}^n$.

Now, define

$$u(x) = \big(\sup \{ w(x) : w \in \mathcal{F} \} \big)^* \quad \text{for } x \in \mathbb{R}^n. \tag{3.2}$$

By construction,

$$u \in \mathrm{USC}(\mathbb{R}^n), \quad \phi \leq u \leq \psi, \quad \text{and} \quad \lim_{|x|\to\infty} (u-\phi)(x) = 0.$$

In particular,

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1+|x|)^{n+2s}}\, dx < \infty.$$

Moreover, since $u - \psi$ is upper semicontinuous in $\mathbb{R}^n$, we have that

$$\text{the noncoincidence set } \{u < \psi\} \text{ is open.} \tag{3.3}$$

First, we will show that u, as defined in (3.2), is in the class $\mathcal{F}$, see Lemma 3.4. We start with two lemmas.

Lemma 3.1. *Let $w_1, w_2 \in \mathcal{F}$. Then,*

$$w(x) = \max\{w_1(x), w_2(x)\} \in \mathcal{F}.$$

Proof. Evidently, $w \in \mathrm{USC}(\mathbb{R}^n)$, $w \leq \psi$, and $\lim_{|x|\to\infty}(w-\phi)(x) \leq 0$. Let P be a C^2 function touching w from above at x_0 in $\mathcal{N}$. Without loss of generality, $P(x_0) = w(x_0) = w_1(x_0)$, so P also touches w_1 from above at x_0 in $\mathcal{N}$. Let

$$\vartheta(x) = \begin{cases} P(x) & \text{if } x \in \mathcal{N} \\ w(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N} \end{cases} \quad \text{and} \quad \vartheta_1(x) = \begin{cases} P(x) & \text{if } x \in \mathcal{N} \\ w_1(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}. \end{cases}$$

Observe that $\vartheta(x_0) = \vartheta_1(x_0)$ and $\vartheta \geq \vartheta_1$ in $\mathbb{R}^n$, from which it follows that $\delta(\vartheta, x_0, y) \geq \delta(\vartheta_1, x_0, y)$ for any $y \in \mathbb{R}^n$. Therefore, given any matrix $A \in \mathcal{M}$,

$$L^s_A \vartheta(x_0) \geq L^s_A \vartheta_1(x_0) \geq \mathcal{D}_s \vartheta_1(x_0) \geq \vartheta_1(x_0) - \phi(x_0) = \vartheta(x_0) - \phi(x_0).$$

Thus, $\mathcal{D}_s \vartheta(x_0) \geq \vartheta(x_0) - \phi(x_0)$, and w is a subsolution to $\mathcal{D}_s w \geq w - \phi$ in $\mathbb{R}^n$. □

Lemma 3.2. *Let u be as in* (3.2) *and let P be a C^2 function touching u from above at x_0 in $\mathcal{N}$. Given any open neighborhood $\mathcal{N}' \subset\subset \mathcal{N}$ that contains x_0, there exist functions $u_k \in \mathcal{F}$, points $x_k \in \mathcal{N}'$, and constants $d_k > 0$, for $k \geq 1$, such that*

$$u_k \leq u_{k+1}, \quad x_k \to x_0, \quad d_k \searrow 0, \quad u_k(x_k) \to u(x_0),$$

and

$$P_k(y) = P(y) + \frac{|y-x_k|^2}{k} - d_k \quad \textit{touches } u_k \textit{ from above at } x_k \textit{ in } \mathcal{N}'. \tag{3.4}$$

Proof. Fix $x_0 \in \mathbb{R}^n$. The proof is divided into two steps.

– *Step 1.* There exist points y_k and functions $u_k \in \mathcal{F}$ with $u_k \leq u_{k+1}$ such that

$$y_k \to x_0 \quad \text{and} \quad u_k(y_k) \to u(x_0).$$

Indeed, by Remark 2.1, there exists a sequence of points y_k such that

$$y_k \to x_0 \quad \text{and} \quad \overline{w}(y_k) \equiv \sup_{w \in \mathcal{F}} w(y_k) \to u(x_0). \tag{3.5}$$

Let $k \geq 1$. There is a sequence $\{w_{k,j}\}_{j=1}^{\infty} \in \mathcal{F}$ such that

$$w_{k,j}(y_k) \nearrow \overline{w}(y_k) \quad \text{as } j \to \infty. \tag{3.6}$$

In particular, there exists $J(k) > 0$ such that

$$0 \leq \overline{w}(y_k) - w_{k,j}(y_k) < 1/k \quad \text{for every } j \geq J(k). \tag{3.7}$$

Without loss of generality we can let $J(k) < J(k+1)$, for every $k \geq 1$. Define

$$u_k(y) = \max\left\{w_{1,J(1)}(y), \ldots, w_{k,J(k)}(y)\right\} \quad \text{for } y \in \mathbb{R}^n.$$

Then, $u_k \leq u_{k+1}$ and, by Lemma 3.1, $u_k \in \mathcal{F}$ for every $k \geq 1$. Finally, observe that, by the definition of u_k, (3.6), (3.5), and (3.7), as $k \to \infty$,

$$\begin{aligned} |u(x_0) - u_k(y_k)| &\leq |u(x_0) - \overline{w}(y_k)| + \big(\overline{w}(y_k) - u_k(y_k)\big) \\ &\leq |u(x_0) - \overline{w}(y_k)| + (\overline{w}(y_k) - w_{k,J(k)}(y_k)) \to 0. \end{aligned}$$

– *Step 2.* Let $\mathcal{N}' \subset\subset \mathcal{N}$ be any open neighborhood of x_0. Without loss of generality, we can assume that the sequence y_k from Step 1 satisfies $y_k \in \mathcal{N}'$ for all $k \geq 1$. Define

$$d_k = \inf_{\overline{\mathcal{N}'}}(P - u_k).$$

Notice that $d_k \geq 0$ is well defined because $P - u_k$ is lower semicontinuous in $\mathbb{R}^n$. Moreover, $d_k \geq d_{k+1}$ as $u_k \leq u_{k+1} \leq u \leq P$ in $\mathcal{N}$. Also,

$$0 \leq d_k \leq P(y_k) - u_k(y_k) \to P(x_0) - u(x_0) = 0.$$

Let $x_k \in \overline{\mathcal{N}'}$ be such that

$$P(x_k) - u_k(x_k) = d_k. \tag{3.8}$$

The set of points $\{x_k\}_{k=1}^{\infty}$ is bounded, so, after passing to a subsequence, we can assume that $\{x_k\}_{k=1}^{\infty}$ is convergent in $\overline{\mathcal{N}'}$.

Let us show that $x_k \to x_0$. Suppose, to the contrary, that there exists a subsequence $\{x_{k_j}\}_{j=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ such that $x_{k_j} \to x' \in \overline{\mathcal{N}'}$, as $j \to \infty$, with $x' \neq x_0$. Then, as $P > u$ in $\overline{\mathcal{N}'} \setminus \{x_0\}$ and $P - u$ is lower semicontinuous,

$$0 < P(x') - u(x') \leq \liminf_{j \to \infty}(P - u)(x_{k_j}) \leq \lim_{j \to \infty} d_{k_j} = 0,$$

which is a contradiction. Hence, $x_k \to x_0$, as desired.

This and (3.8) imply that $u_k(x_k) \to u(x_0)$. By construction, $P(x_k) - d_k = u_k(x_k)$ and $P - d_k \geq u_k$ in $\mathcal{N}'$. So, $P_k(y)$ as defined in (3.4) is a C^2 function that touches u_k from above at x_k in $\mathcal{N}'$. □

Remark 3.3. In Lemma 3.2, we can modify the definition of $P_k(y)$ in (3.4). Indeed, as the proof above shows, any function of the form

$$P(y) + \varphi(y) - d_k,$$

where φ is a C^2 function such that $\varphi(x_k) = 0$ and $\varphi(y) > 0$ for all $y \in \overline{\mathcal{N}'} \setminus \{x_k\}$, will touch u_k from above at x_k in $\mathcal{N}'$.

Lemma 3.4. *Let u be as in* (3.2)*. Then,*

$$\mathcal{D}_s u \geq u - \phi \quad \text{in } \mathbb{R}^n.$$

In particular,

$$u \leq \bar{u},$$

where $\bar{u}$ is the solution to (1.4)*.*

Proof. Let P be a C^2 function touching u from above at x_0 in $\mathcal{N}$. By Lemma 3.2, there exist functions $u_k \in \mathcal{F}$, points $x_k \in B_r(x_0) \subset\subset \mathcal{N}$ for some $r > 0$, and constants $d_k > 0$ such that $u_k \leq u_{k+1} \leq u$, $d_k \searrow 0$, $u_k(x_k) \to u(x_0)$, and $P_k(y) = P(y) + \frac{1}{k}|y - x_k|^2 - d_k$ touches u_k from above at x_k in $B_r(x_0)$ for $k \geq 1$. Define the test functions

$$\vartheta(x) = \begin{cases} P(x) & \text{if } x \in B_r(x_0) \\ u(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0) \end{cases} \quad \text{and} \quad \vartheta_k(x) = \begin{cases} P_k(x) & \text{if } x \in B_r(x_0) \\ u_k(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

We recall that, by Remark 2.4, it is enough to use ϑ as defined above as a test function for u. Let $A \in \mathcal{M}$ and let Λ_A denote the maximum eigenvalue of A. Then,

$$\begin{aligned}
&c_{n,s}^{-1} L_A^s \vartheta_k(x_k) \\
&\quad = \lim_{\rho \to 0} \int_{B_r(x_0) \setminus B_\rho(x_k)} \frac{\vartheta_k(y) - \vartheta_k(x_k)}{|A^{-1}(y - x_k)|^{n+2s}}\, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{\vartheta_k(y) - \vartheta_k(x_k)}{|A^{-1}(y - x_k)|^{n+2s}}\, dy \\
&\quad \leq \lim_{\rho \to 0} \left[\int_{B_r(x_0) \setminus B_\rho(x_k)} \frac{P(y) - P(x_k)}{|A^{-1}(y - x_k)|^{n+2s}}\, dy + \frac{1}{k} \int_{B_r(x_0) \setminus B_\rho(x_k)} \frac{|y - x_k|^2}{|A^{-1}(y - x_k)|^{n+2s}}\, dy \right] \\
&\qquad + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{u(y) - P(x_k)}{|A^{-1}(y - x_k)|^{n+2s}}\, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{d_k}{|A^{-1}(y - x_k)|^{n+2s}}\, dy \\
&\quad \leq c_{n,s}^{-1} L_A^s \vartheta(x_k) + \Lambda_A^{n+2s} \left[\frac{1}{k} \int_{B_r(x_0)} \frac{1}{|y - x_k|^{n+2s-2}}\, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{d_k}{|y - x_k|^{n+2s}}\, dy \right] \\
&\quad \leq c_{n,s}^{-1} L_A^s \vartheta(x_k) + C\big(k^{-1} + d_k\big),
\end{aligned}$$

where $C = C(n, s, \Lambda_A, r) > 0$ is independent of k. Here, we have used that

$$\int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{1}{|y - x_k|^{n+2s}}\, dy \to \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{1}{|y - x_0|^{n+2s}}\, dy$$

and

$$\int_{B_r(x_0)} \frac{1}{|y - x_k|^{n+2s-2}}\, dy \to \int_{B_r(x_0)} \frac{1}{|y - x_0|^{n+2s-2}}\, dy,$$

as $k \to \infty$. Hence, as u_k is a subsolution,

$$\vartheta_k(x_k) - \phi(x_k) \leq \mathcal{D}_s \vartheta_k(x_k) \leq L_A^s \vartheta(x_k) + C\big(k^{-1} + d_k\big).$$

Notice that $\vartheta_k(x_k) = P_k(x_k) \to P(x_0) = \vartheta(x_0)$ as $k \to \infty$. Together with Lemma 2.8, this implies that

$$\vartheta(x_0) - \phi(x_0) \leq L_A^s \vartheta(x_0).$$

Since $A \in \mathcal{M}$ was arbitrary, we obtain $\vartheta(x_0) - \phi(x_0) \leq \mathcal{D}_s \vartheta(x_0)$, which means that u is a subsolution to $\mathcal{D}_s u \geq u - \phi$.

We have already seen that $\lim_{|x| \to \infty}(u - \phi)(x) = 0$. Thus, the comparison principle (Theorem 2.6) implies that $u \leq \bar{u}$. □

With Lemma 3.4 in hand, we can prove that the contact set $\{u = \psi\}$ is compact and that u is Lipschitz and semiconcave with constants no larger than those of ϕ and ψ.

Lemma 3.5. *Let u be as in* (3.2). *Then,*

$$\{u = \psi\} \text{ is compact.}$$

Proof. We know that $u \leq \psi$ and that the noncoincidence set $\{u < \psi\}$ is open, see (3.3). Therefore, the contact set $\{u = \psi\}$ is closed. On the other hand, by Lemma 3.4, $\{\bar{u} < \psi\} \subset \{u < \psi\}$, which implies that $\{u = \psi\} \subset \mathcal{K}$. Hence, the contact set is compact. □

Recall the definition of M_1 and M_2 from the statement of Theorem 1.1.

Lemma 3.6. *Let u be as in* (3.2). *Then, u is Lipschitz continuous and semiconcave with*

$$[u]_{\mathrm{Lip}(\mathbb{R}^n)} \leq M_1 \quad \text{and} \quad \mathrm{SC}(u) \leq M_2.$$

Proof. Given any $h \in \mathbb{R}^n$, let us first show that

$$w(x) = u(x+h) - M_1|h| \in \mathcal{F}. \tag{3.9}$$

Indeed, $w \in \mathrm{USC}(\mathbb{R}^n)$,

$$\lim_{|x|\to\infty} (w - \phi)(x) = \lim_{|x|\to\infty} \Big[\big(u(x+h) - u(x)\big) + \big(u(x) - \phi(x)\big)\Big] - M_1|h| \leq 0,$$

and since $-M_1|h| \leq \psi(x) - \psi(x+h)$ and $u \leq \psi$,

$$w(x) = u(x+h) - M_1|h| \leq u(x+h) - \psi(x+h) + \psi(x) \leq \psi(x).$$

Finally, as $\mathcal{D}_s$ is translation invariant, $\mathcal{D}_s c = 0$ for any constant c, and $\phi(x+h) - \phi(x) \leq M_1|h|$, we find that

$$\begin{aligned} \mathcal{D}_s w(x) &= \mathcal{D}_s(\tau_h u)(x) = (\mathcal{D}_s u)(x+h) \geq u(x+h) - \phi(x+h) \\ &= (u(x+h) - M_1|h|) - \phi(x+h) + M_1|h| \\ &\geq w(x) - \phi(x), \end{aligned}$$

in the viscosity sense. Thus, (3.9) is proved. Now, by the maximality of u in $\mathcal{F}$, $w \leq u$, which means that

$$u(x+h) - u(x) \leq M_1|h|.$$

Since x and h above are arbitrary, we conclude that $[u]_{\mathrm{Lip}(\mathbb{R}^n)} \leq M_1$.

Given any $h \in \mathbb{R}^n$, let us first see that

$$w(x) = \frac{u(x+h) + u(x-h) - M_2|h|^2}{2} \in \mathcal{F}. \tag{3.10}$$

Indeed, $w \in \mathrm{USC}$, and since $\delta(\phi, x, h) \leq M_2|h|^2$,

$$\begin{aligned} (w - \phi)(x) &= \frac{u(x+h) + u(x-h)}{2} - \frac{M_2|h|^2}{2} - \phi(x) \\ &= \frac{(u-\phi)(x+h) + (u-\phi)(x-h)}{2} + \frac{\delta(\phi, x, h) - M_2|h|^2}{2} \\ &\leq \frac{(u-\phi)(x+h) + (u-\phi)(x-h)}{2} \to 0 \end{aligned}$$

as $|x| \to \infty$. Also, $u \leq \psi$ and $\delta(\psi, x, h) \leq M_2|h|^2$, which implies that

$$w(x) = \frac{u(x+h) + u(x-h)}{2} - \frac{M_2|h|^2}{2} \leq \frac{\psi(x+h) + \psi(x-h)}{2} - \frac{M_2|h|^2}{2} \leq \psi(x).$$

Finally, using the inequality $M_2|h|^2 - \delta(\phi, x, h) \geq 0$,

$$\begin{aligned} \mathcal{D}_s w(x) &\geq \frac{1}{2}\mathcal{D}_s(\tau_h u)(x) + \frac{1}{2}\mathcal{D}_s(\tau_{-h} u)(x) \\ &= \frac{1}{2}\mathcal{D}_s u(x+h) + \frac{1}{2}\mathcal{D}_s u(x-h) \end{aligned}$$

$$\geq \frac{(u-\phi)(x+h)+(u-\phi)(x-h)}{2}$$
$$= w(x) + \frac{M_2|h|^2 - \delta(\phi,x,h)}{2} - \phi(x) \geq w(x) - \phi(x),$$

in the viscosity sense. Thus, (3.10) is proved. By the maximality of u in $\mathcal{F}$, we have that $w \leq u$. Hence,

$$u(x+h) + u(x-h) - 2u(x) \leq M_2|h|^2$$

or, equivalently, u is semiconcave and $\mathrm{SC}(u) \leq M_2$. □

The semiconcavity of u permits us to compute $\mathcal{D}_s u(x)$ in the classical sense, see Lemma 2.5. We use this to show that $\mathcal{D}_s u(x)$ is bounded from above.

Lemma 3.7. *Let u be as in (3.2). Then, $\mathcal{D}_s u(x)$ can be computed in the classical sense and*

$$0 \leq \mathcal{D}_s u(x) \leq C\big(1 + \|u-\phi\|_{L^\infty(\mathbb{R}^n)}\big) \quad \textit{for every } x \in \mathbb{R}^n,$$

for some constant $C = C(n,s,a,\epsilon,M_2) > 0$.

Proof. As u is semiconcave on $\mathbb{R}^n$ (see Lemma 3.6), it can be touched from above by a C^2 function at every point $x \in \mathbb{R}^n$. Thus, Lemmas 3.4 and 2.5 imply that $\mathcal{D}_s u(x)$ can be computed classically and $\mathcal{D}_s u(x) \geq u(x) - \phi(x) \geq 0$ for every $x \in \mathbb{R}^n$. Since, for any $x \in \mathbb{R}^n$, we have $\delta(u,x,y)/|y|^{n+2s} \in L^1(\mathbb{R}^n)$ and $\delta(\phi,x,y) \geq 0$, we can estimate

$$\begin{aligned}
\mathcal{D}_s u(x) &\leq -(-\Delta)^s u(x) \\
&= c_{n,s}\int_{B_1} \frac{\delta(u,x,y)}{|y|^{n+2s}}\,dy + c_{n,s}\int_{\mathbb{R}^n\setminus B_1} \frac{\delta(u,x,y)}{|y|^{n+2s}}\,dy \\
&\leq c_{n,s}\int_{B_1} \frac{\mathrm{SC}(u)|y|^2}{|y|^{n+2s}}\,dy + c_{n,s}\int_{\mathbb{R}^n\setminus B_1} \frac{\delta(u-\phi,x,y)}{|y|^{n+2s}}\,dy + c_{n,s}\int_{\mathbb{R}^n\setminus B_1} \frac{\delta(\phi,x,y)}{|y|^{n+2s}}\,dy \\
&\leq C\big(M_2 + \|u-\phi\|_{L^\infty(\mathbb{R}^n)} - (-\Delta)^s\phi(x)\big) \\
&\leq C\big(1 + \|u-\phi\|_{L^\infty(\mathbb{R}^n)}\big).
\end{aligned}$$

In the last inequality, we have used that

$$0 \leq -(-\Delta)^s\phi \leq M_0 \quad \text{in } \mathbb{R}^n, \tag{3.11}$$

for some constant $M_0 = M_0(n,s,a,\epsilon) > 0$, see [1, eq. (6.8)]. □

Next we need to consider the obstacle problem (1.1) for the truncated fractional Monge–Ampère operator defined in subsection 2.3.

Theorem 3.8. *For any $\varepsilon > 0$, there exists a unique classical solution u_ε to the obstacle problem*

$$\begin{cases}
\mathcal{D}_s^\varepsilon u_\varepsilon \geq u_\varepsilon - \phi & \textit{in } \mathbb{R}^n \\
u_\varepsilon \leq \psi & \textit{in } \mathbb{R}^n \\
\mathcal{D}_s^\varepsilon u_\varepsilon = u_\varepsilon - \phi & \textit{in } \{u_\varepsilon < \psi\} \\
\lim\limits_{|x|\to\infty} (u_\varepsilon - \phi)(x) = 0. &
\end{cases}$$

Moreover, u_ε is Lipschitz and semiconcave with constants no larger than M_1 and M_2, respectively.

Proof. Fix $\varepsilon > 0$. Parallel to (3.1), we define the class

$$\mathcal{F}_\varepsilon = \Big\{ w \in \mathrm{USC}(\mathbb{R}^n) : \mathcal{D}_s^\varepsilon w \geq w - \phi \text{ in } \mathbb{R}^n,\ w \leq \psi, \text{ and } \lim_{|x|\to\infty}(w-\phi)(x) \leq 0 \Big\}.$$

Then, $\phi \in \mathcal{F}_\varepsilon$. By replacing u by u_ε and $\mathcal{D}_s$ by $\mathcal{D}_s^\varepsilon$ in the arguments of Lemmas 3.1–3.4 and Lemma 3.6, we deduce that

$$u_\varepsilon(x) = \big(\sup\{w(x) : w \in \mathcal{F}_\varepsilon\}\big)^* \quad \text{for } x \in \mathbb{R}^n$$

is the largest function in $\mathcal{F}_\varepsilon$, $\phi \leq u_\varepsilon \leq \psi$, $\lim_{|x|\to\infty}(u_\varepsilon - \phi)(x) = 0$,

$$[u_\varepsilon]_{\mathrm{Lip}(\mathbb{R}^n)} \leq M_1 \quad \text{and} \quad \mathrm{SC}(u_\varepsilon) \leq M_2, \tag{3.12}$$

and the noncoincidence set $\{u_\varepsilon < \psi\}$ is open.

It remains to prove that

$$\mathcal{D}_s^\varepsilon u_\varepsilon = u_\varepsilon - \phi \quad \text{in } \{u_\varepsilon < \psi\}.$$

We argue by contradiction. Specifically, we will show that if $\mathcal{D}_s^\varepsilon u_\varepsilon = u_\varepsilon - \phi$ fails in the open set $\{u_\varepsilon < \psi\}$, then u_ε is not maximal in $\mathcal{F}_\varepsilon$. To this end, let $x_0 \in \{u_\varepsilon < \psi\}$ and P be a C^2 function touching u_ε from below at x_0 in $\overline{\mathcal{N}}$ such that for

$$\vartheta(x) = \begin{cases} P(x) & \text{for } x \in \mathcal{N} \\ u_\varepsilon(x) & \text{for } x \in \mathbb{R}^n \setminus \mathcal{N}, \end{cases}$$

we have

$$\mathcal{D}_s^\varepsilon \vartheta(x_0) > \vartheta(x_0) - \phi(x_0).$$

Recall that $\mathcal{D}_s^\varepsilon \vartheta$ is continuous in $\mathcal{N}$ (see Lemma 2.8). Hence, given

$$0 < \tau < \mathcal{D}_s^\varepsilon \vartheta(x_0) - (\vartheta(x_0) - \phi(x_0)),$$

there exists a ball $B_r(x_0) \subset\subset \mathcal{N} \cap \{u_\varepsilon < \psi\}$ such that

$$\mathcal{D}_s^\varepsilon \vartheta(z) \geq \vartheta(z) - \phi(z) + \tau \quad \text{for every } z \in B_r(x_0). \tag{3.13}$$

Next, we lift P in $\mathcal{N}$ by a small amount $d > 0$ (to be fixed) so that $\{u_\varepsilon < P + d\} \subset\subset B_r(x_0)$ and $\{P + d < \psi\} \subset \{u_\varepsilon < P + d\}$. We then set

$$u_\varepsilon'(x) = \begin{cases} P(x) + d & \text{if } x \in \{u_\varepsilon < P + d\} \\ u_\varepsilon(x) & \text{otherwise,} \end{cases}$$

and notice that u_ε' is continuous, $u_\varepsilon' \geq u_\varepsilon$ in $\mathbb{R}^n$, and $\lim_{|x|\to\infty}(u_\varepsilon' - \phi)(x) = 0$. If we can show that u_ε' is a subsolution to $\mathcal{D}_s^\varepsilon w = w - \phi$, then u_ε is not maximal in $\mathcal{F}_\varepsilon$ because we constructed u_ε' in such a way that

$$u_\varepsilon'(x_0) = P(x_0) + d > P(x_0) = u_\varepsilon(x_0).$$

Therefore, we now prove that

$$u_\varepsilon' \text{ is a subsolution to } \mathcal{D}_s^\varepsilon w = w - \phi \text{ in } \mathbb{R}^n. \tag{3.14}$$

Let P' be a C^2 function touching u_ε' from above at x' in $\mathcal{N}'$. We have two cases to consider.

– *Case 1.* $x' \in \{u_\varepsilon' = u_\varepsilon\}$. Since $\delta(u_\varepsilon', x', y) \geq \delta(u_\varepsilon, x', y)$, by Lemma 2.5 (which is also valid for the uniformly elliptic case), we see that

$$\mathcal{D}_s^\varepsilon u_\varepsilon'(x') \geq \mathcal{D}_s^\varepsilon u_\varepsilon(x') \geq u_\varepsilon(x') - \phi(x') = u_\varepsilon'(x') - \phi(x'),$$

and (3.14) follows.

– *Case 2.* $x' \in \{u_\varepsilon' > u_\varepsilon\}$. Define

$$\vartheta'(x) = \begin{cases} P'(x) & \text{if } x \in \mathcal{N}' \\ u_\varepsilon'(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}'. \end{cases}$$

Remark 2.4 allows us to assume that $\mathcal{N}' \subset \{u'_\varepsilon > u_\varepsilon\} = \{u_\varepsilon < P + d\} \subset\subset B_r(x_0)$. Observe that $P' - d \geq u'_\varepsilon - d = P$ in $\mathcal{N}'$ and $P'(x') - d = P(x')$. Then,

$$\begin{aligned} c_{n,s}^{-1} L_A^s \vartheta'(x') &= \lim_{\rho\to 0} \int_{\mathcal{N}'\setminus B_\rho(x')} \frac{P'(y) - u'_\varepsilon(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy + \int_{\mathbb{R}^n\setminus\mathcal{N}'} \frac{u'_\varepsilon(y) - u'_\varepsilon(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy \\ &= \lim_{\rho\to 0} \int_{\mathcal{N}'\setminus B_\rho(x')} \frac{(P'(y)-d) - (P'(x')-d)}{|A^{-1}(y-x')|^{n+2s}}\,dy + \mathrm{I} \\ &\geq \lim_{\rho\to 0} \int_{\mathcal{N}'\setminus B_\rho(x')} \frac{P(y) - P(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy + \mathrm{I}. \end{aligned} \tag{3.15}$$

We estimate the integral I from below. Since $u'_\varepsilon \geq u_\varepsilon$ in $\mathbb{R}^n$, $u'_\varepsilon(x') = P'(x') = P(x') + d$, and $u'_\varepsilon \geq P + d$ in $\mathcal{N} \setminus \mathcal{N}'$, we find that

$$\begin{aligned} \mathrm{I} &= \int_{\mathbb{R}^n\setminus\mathcal{N}} \frac{u'_\varepsilon(y) - u'_\varepsilon(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy + \int_{\mathcal{N}\setminus\mathcal{N}'} \frac{u'_\varepsilon(y) - u'_\varepsilon(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy \\ &\geq \int_{\mathbb{R}^n\setminus\mathcal{N}} \frac{u_\varepsilon(y) - P(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy - \int_{\mathbb{R}^n\setminus\mathcal{N}} \frac{d}{|A^{-1}(y-x')|^{n+2s}}\,dy \\ &\quad + \int_{\mathcal{N}\setminus\mathcal{N}'} \frac{P(y) - P(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy. \end{aligned} \tag{3.16}$$

From (3.15), (3.16) and the definition of ϑ, we get

$$\begin{aligned} c_{n,s}^{-1} L_A^s \vartheta'(x') &\geq \lim_{\rho\to 0} \int_{\mathcal{N}\setminus B_\rho(x')} \frac{P(y) - P(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy + \int_{\mathbb{R}^n\setminus\mathcal{N}} \frac{u_\varepsilon(y) - P(x')}{|A^{-1}(y-x')|^{n+2s}}\,dy \\ &\quad - d\int_{\mathbb{R}^n\setminus\mathcal{N}} \frac{1}{|A^{-1}(y-x')|^{n+2s}}\,dy \\ &= c_{n,s}^{-1} L_A^s \vartheta(x') - Cd, \end{aligned}$$

where $C = C(n, s, \varepsilon, \mathcal{N}) > 0$ is independent of A. By taking the infimum over all $A \in \mathcal{M}_\varepsilon$ above and using (3.13), we deduce that

$$\mathcal{D}_s^\varepsilon \vartheta'(x') \geq \mathcal{D}_s^\varepsilon \vartheta(x') - Cd \geq \vartheta(x') - \phi(x') + \tau - Cd = \vartheta'(x') - \phi(x') + \tau - (C+1)d.$$

Thus, by choosing $d > 0$ sufficiently small, it follows that

$$\mathcal{D}_s^\varepsilon \vartheta'(x') \geq \vartheta'(x') - \phi(x').$$

This completes the proof of (3.14) and the theorem. □

Let u_ε and $\mathcal{F}_\varepsilon$ be as in Theorem 3.8 and its proof. First, notice that if $\varepsilon_0 \geq \varepsilon$, then $\mathcal{D}_s^{\varepsilon_0} u_\varepsilon \geq \mathcal{D}_s^\varepsilon u_\varepsilon$. In particular, $u_\varepsilon \in \mathcal{F}_{\varepsilon_0}$. Hence, by the maximality of u_{ε_0} in $\mathcal{F}_{\varepsilon_0}$, we have that $u_{\varepsilon_0} \geq u_\varepsilon$. In other words, the sequence of functions u_ε is decreasing as $\varepsilon \searrow 0$. Let

$$u_0(x) = \inf_{\varepsilon>0} u_\varepsilon(x) \quad \text{for } x \in \mathbb{R}^n. \tag{3.17}$$

Then, u_0 is well-defined because $\phi \leq u_\varepsilon \leq \psi$ for every $\varepsilon > 0$. Clearly,

$$\phi \leq u_0 \leq \psi \quad \text{and} \quad \lim_{|x|\to\infty} (u_0 - \phi)(x) = 0.$$

Moreover, (3.12) and Arzelà–Ascoli's theorem imply that u_0 is the local uniform (decreasing) limit of u_ε and that u_0 is Lipschitz continuous with $[u_0]_{\mathrm{Lip}(\mathbb{R}^n)} \leq M_1$.

The following is one of the crucial, most delicate estimates we need.

Lemma 3.9. *Let u_0 be as in (3.17). Then,*

$$u_0 > \phi.$$

Proof. Let us argue by contradiction. Suppose that there is a point x_0 such that $u_0(x_0) = \phi(x_0)$. Then, as $\phi < \psi$ in $\mathbb{R}^n$, we have $x_0 \in \{u_0 < \psi\}$. Since $\phi \in C^{2,\sigma}(\mathbb{R}^n)$ is strictly convex in compact sets and asymptotically close to a cone at infinity, we can find a function $\varphi \in C^{2,\sigma}(\mathbb{R}^n)$ that is also strictly convex in compact sets, asymptotically close to a cone at infinity, and touches both u_0 and ϕ from below in $\overline{B_r(x_0)}$ at x_0 for some $r > 0$. As u_0 is the local uniform limit of u_ε, there exist points $x_\varepsilon \in B_r(x_0)$ such that $x_\varepsilon \to x_0$ and u_ε can be touched from below at x_ε in $B_r(x_0)$ by

$$\varphi_\varepsilon(x) = \varphi(x) - \varepsilon\omega(|x - x_\varepsilon|) + d_\varepsilon \quad \text{for } x \in \mathbb{R}^n. \tag{3.18}$$

Here, $d_\varepsilon \searrow 0$ and $\omega = \omega(t)$ is convex, strictly increasing in $[0, r)$, smooth in $(0, r)$, linear in $\mathbb{R} \setminus [0, r)$, $\omega(0^+) = \omega'(0^+) = 0$, and such that φ_ε is strictly convex in compact sets. Because φ_ε is convex and touches the supersolution u_ε from below at x_ε,

$$0 \leq \mathcal{D}_s\varphi_\varepsilon(x_\varepsilon) \leq \mathcal{D}_s^\varepsilon\varphi_\varepsilon(x_\varepsilon) \leq \varphi_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon). \tag{3.19}$$

As the sets $\{u_\varepsilon < \psi\}$ are increasing and $x_\varepsilon \to x_0$, we have $x_\varepsilon \in \{u_\varepsilon < \psi\}$ for all $\varepsilon > 0$ sufficiently small. Moreover,

$$u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0.$$

This and (3.19) imply that

$$0 \leq \mathcal{D}_s\varphi_\varepsilon(x_\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0. \tag{3.20}$$

Using this last inequality, we will prove that there exists a direction $\mathrm{e}_0 \in \mathbb{S}^{n-1}$ such that

$$-(-\Delta)^s_{\mathrm{e}_0}\varphi(x_0) = \frac{c_{n,s}}{2}\int_{\mathbb{R}} \frac{\delta(\varphi, x_0, t\mathrm{e}_0)}{|t|^{1+2s}}\,dt \leq 0. \tag{3.21}$$

This clearly contradicts the convexity of the nonconstant function φ. In turn, $u_0 > \phi$, as desired.

To deduce (3.21), suppose, to the contrary, that

$$-(-\Delta)^s_{\mathrm{e}}\varphi(x_0) \geq \mu > 0 \quad \text{for all } \mathrm{e} \in \mathbb{S}^{n-1}. \tag{3.22}$$

Since the family $\{-(-\Delta)^s_{\mathrm{e}}\varphi\}_{\mathrm{e}\in\mathbb{S}^{n-1}}$ is equicontinuous (see (2.5)),

$$\inf_{\mathrm{e}\in\mathbb{S}^{n-1}} \big\{-(-\Delta)^s_{\mathrm{e}}\varphi(x_\varepsilon)\big\} \geq \frac{\mu}{2} > 0 \quad \text{for all } \varepsilon \text{ sufficiently small.}$$

The function $\omega \equiv \omega(|\cdot|)$ in (3.18) is radially symmetric and convex. Hence, $(-\Delta)^s_{\mathrm{e}}\omega(0)$ is a negative constant independent of $\mathrm{e} \in \mathbb{S}^{n-1}$. Therefore, we can ensure that $\varepsilon(-\Delta)^s_{\mathrm{e}}\omega(0) \geq -\mu/4$ provided ε is sufficiently small, independently of the direction $\mathrm{e} \in \mathbb{S}^{n-1}$. Collecting these last two facts and (3.18), we deduce that

$$-(-\Delta)^s_{\mathrm{e}}\varphi_\varepsilon(x_\varepsilon) = -(-\Delta)^s_{\mathrm{e}}\varphi(x_\varepsilon) + \varepsilon(-\Delta)^s_{\mathrm{e}}\omega(0) \geq \frac{\mu}{4} > 0, \tag{3.23}$$

uniformly in $\mathrm{e} \in \mathbb{S}^{n-1}$ and for all ε sufficiently small. It is show in the proof of [1, Proposition 3.5] that an estimate of the form (3.23) readily yields the existence of a positive constant $\tau = \tau(n, s, \mu, [\varphi]_{\mathrm{Lip}(\mathbb{R}^n)}, \mathrm{SC}(\varphi))$ such that $\mathcal{D}_s\varphi_\varepsilon(x_\varepsilon) \geq \tau$. This estimate is uniform in ε, a contradiction to (3.20). Thus, (3.22) cannot hold. In other words, there are directions $\mathrm{e}_k \in \mathbb{S}^{n-1}$ such that

$$-(-\Delta)^s_{\mathrm{e}_k}\varphi(x_0) = \frac{c_{n,s}}{2}\int_{\mathbb{R}} \frac{\delta(\varphi, x_0, t\mathrm{e}_k)}{|t|^{1+2s}}\,dt \leq \frac{1}{k}, \tag{3.24}$$

for each $k \geq 1$. The compactness of $\mathbb{S}^{n-1}$ allows us to assume, without loss of generality, that $\mathrm{e}_k \to \mathrm{e}_0$ for some $\mathrm{e}_0 \in \mathbb{S}^{n-1}$, as $k \to \infty$. The continuity of φ gives

$$\frac{\delta(\varphi, x_0, t\mathrm{e}_k)}{|t|^{1+2s}} \to \frac{\delta(\varphi, x_0, t\mathrm{e}_0)}{|t|^{1+2s}},$$

as $k \to \infty$. Since φ is convex and has linear growth at infinity,

$$0 \leq \frac{\delta(\varphi, x_0, te_k)}{|t|^{1+2s}} \leq \frac{\min\{2[\varphi]_{\mathrm{Lip}(\mathbb{R}^n)}|t|, \mathrm{SC}(\varphi)|t|^2\}}{|t|^{n+2s}} \in L^1(\mathbb{R}),$$

uniformly in $k \geq 1$. Thus, we can apply the dominated convergence theorem to (3.24) to get

$$-(-\Delta)^s_{\mathrm{e}_0}\varphi(x_0) = -\lim_{k\to\infty}(-\Delta)^s_{\mathrm{e}_k}\varphi(x_0) \leq 0,$$

as desired. □

Lemma 3.10. *Let u_0 be as in* (3.17)*. Then,*

$$\mathcal{D}_s u_0 \leq u_0 - \phi \quad \text{in } \{u_0 < \psi\}.$$

Proof. Let $x_0 \in \{u_0 < \psi\}$ be a point at which u_0 can be touched from below by a C^2 function in a neighborhood $\mathcal{N} \subset\subset \{u_0 < \psi\}$. As u_ε decreases locally uniformly to u_0, we can find a sequence of points $x_\varepsilon \to x_0$ and C^2 functions P_ε that touch u_ε from below at x_ε in a common neighborhood $\mathcal{N}' \subset\subset \mathcal{N}$. By Lemma 3.9, we have $u_\varepsilon \geq u_0 > \phi$. Then, by Theorem 2.7, there exists $\lambda > 0$ such that

$$\mathcal{D}_s u_0(x_0) = \mathcal{D}^\lambda_s u_0(x_0) \quad \text{and} \quad \mathcal{D}^\varepsilon_s u_\varepsilon(x_\varepsilon) = \mathcal{D}^\lambda_s u_\varepsilon(x_\varepsilon)$$

for every ε such that $\varepsilon < \lambda$. Now, since u_ε is a supersolution in $\{u_\varepsilon < \psi\}$ (see Theorem 3.8) that can be touched from below by a C^2 function at x_ε in $\mathcal{N}'$, we can apply Lemma 2.5. Consequently,

$$\mathcal{D}^\lambda_s u_\varepsilon(x_\varepsilon) = \mathcal{D}^\varepsilon_s u_\varepsilon(x_\varepsilon) \leq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon). \tag{3.25}$$

By Lemma 2.9,

$$\lim_{\varepsilon\to 0}\mathcal{D}^\lambda_s u_\varepsilon(x_\varepsilon) = \mathcal{D}^\lambda_s u_0(x_0).$$

Thus, by letting $\varepsilon \to 0$ in (3.25),

$$\mathcal{D}_s u_0(x_0) = \mathcal{D}^\lambda_s u_0(x_0) \leq u_0(x_0) - \phi(x_0).$$

□

With the following result we are able to conclude the proof of Theorem 1.1.

Lemma 3.11. *Let u and u_0 be as in* (3.2) *and* (3.17)*, respectively. Then,*

$$u = u_0.$$

Proof. Let us show that $u_0 \in \mathcal{F}$. Let $A \in \mathcal{M}$. If ε is smaller than the minimum eigenvalue of A, then $A \in \mathcal{M}_\varepsilon$, and, as such, we have $L^s_A u_\varepsilon \geq \mathcal{D}^\varepsilon_s u_\varepsilon \geq u_\varepsilon - \phi$ in $\mathbb{R}^n$. Therefore, by Lemma 2.9, we find that $L^s_A u_0 \geq u_0 - \phi$ in $\mathbb{R}^n$. As $A \in \mathcal{M}$ was arbitrary, it follows that

$$\mathcal{D}_s u_0 \geq u_0 - \phi \quad \text{in } \mathbb{R}^n.$$

Therefore, $u_0 \in \mathcal{F}$ and $u_0 \leq u$. For the opposite inequality, observe that

$$\mathcal{D}^\varepsilon_s u \geq \mathcal{D}_s u \geq u - \phi \quad \text{in } \mathbb{R}^n.$$

Whence, $u \in \mathcal{F}_\varepsilon$ for all $\varepsilon > 0$, and by the maximality of u_ε in $\mathcal{F}_\varepsilon$, $u \leq u_\varepsilon$ for all $\varepsilon > 0$. Thus, from the definition of u_0, we determine that $u \leq u_0$. □

Proof of Theorem 1.1. The proof follows from Lemmas 3.4–3.7 and Lemmas 3.9–3.11. □

4. Proof of Theorem 1.2

We first prove Theorem 1.2(1), that is, the local Hölder continuity of ∇u in the noncoincidence set.

Proof of Theorem 1.2(1). Let $\mathcal{O}$ and $\mathcal{O}_\delta$ be as in the statement. Since $u > \phi$ in $\mathbb{R}^n$, by Theorem 2.7, there exists $\lambda = \lambda(n, s, \inf_{\mathcal{O}_\delta}(u-\phi), M_1, M_2) > 0$ such that

$$\mathcal{D}_s u(x) = \mathcal{D}_s^\lambda u(x) \quad \text{for all } x \in \mathcal{O}_\delta.$$

For any $A \in \mathcal{M}_\lambda$, let

$$b_A(x) = L_A^s \phi(x) - (u-\phi)(x).$$

Since u and ϕ are Lipschitz, $\phi \in C^{2,\sigma}(\mathbb{R}^n)$ and satisfies (3.11), and $\lim_{|x|\to\infty}(u-\phi)(x) = 0$, we deduce that

$$\sup_{A\in\mathcal{M}_\lambda} \left\{ \|b_A\|_{L^\infty(\mathcal{O}_\delta)} + [b_A]_{\mathrm{Lip}(\mathcal{O}_\delta)} \right\} \leq C_0, \tag{4.1}$$

where $C_0 = C_0(n, s, \lambda, M_0, M_1, \mathrm{SC}(\phi)) > 0$. Let $w = u - \phi$. We have

$$L_A^s w + b_A(x) = L_A^s u - (u-\phi) \quad \text{in } \mathcal{O}_\delta.$$

By taking the infimum over all $A \in \mathcal{M}_\lambda$ above, we see that w solves

$$\begin{cases} \inf\limits_{A\in\mathcal{M}_\lambda} \left\{ L_A^s w + b_A(x) \right\} = 0 & \text{in } \mathcal{O}_\delta \\ w = u - \phi \in L^\infty(\mathbb{R}^n), \end{cases}$$

with b_A satisfying the uniform estimate (4.1). From Theorem 1.3(b) in [13], the conclusion follows. $\square$

Next we prove Theorem 1.2(2), which establishes that ∇u is Hölder continuous across the free boundary. Recall that the contact set $\{u = \psi\}$ is compact, see Lemma 3.5. Let $\mathcal{B}$ be as in the statement. Set $C\mathcal{B} = \{Cx : x \in \mathcal{B}\}$ with $C > 0$. By Theorem 2.7, there exists

$$\lambda = \lambda(n, s, \inf_{4\mathcal{B}}(u-\phi), M_1, M_2) > 0 \tag{4.2}$$

such that

$$\mathcal{D}_s u = \mathcal{D}_s^\lambda u \quad \text{in } 4\mathcal{B}. \tag{4.3}$$

Define

$$c_A(x) = (u-\phi)(x) - L_A^s \psi(x). \tag{4.4}$$

Since $\sup_{A\in\mathcal{M}_\lambda} [L_A^s \psi]_{\mathrm{Lip}(\mathbb{R}^n)} < \infty$ and $u, \phi \in \mathrm{Lip}(\mathbb{R}^n)$, up to dividing by a constant depending on λ, we can assume that

$$\sup_{A\in\mathcal{M}_\lambda} [c_A]_{\mathrm{Lip}(\mathbb{R}^n)} = 1. \tag{4.5}$$

We subtract the obstacle and let v be as in (1.8). For any $A \in \mathcal{M}$, we have

$$L_A^s v + c_A(x) = -L_A^s u + (u-\phi),$$

so that

$$\sup_{A\in\mathcal{M}_\lambda} \left\{ L_A^s v + c_A(x) \right\} = -\mathcal{D}_s^\lambda u + (u-\phi).$$

Therefore, from (4.3) and up to dividing v by a normalizing constant depending on λ, we get

$$\begin{cases} v \geq 0 & \text{in } \mathbb{R}^n \\ D^2 v(x) \geq -\operatorname{Id} & \text{for a.e. } x \in \mathbb{R}^n \\ \sup_{A\in\mathcal{M}_\lambda} \left\{L_A^s v(x) + c_A(x)\right\} = 0 & \text{in } \{v>0\}\cap 4\mathcal{B} \\ \sup_{A\in\mathcal{M}_\lambda} \left\{L_A^s v(x) + c_A(x)\right\} \leq 0 & \text{in } \mathbb{R}^n \\ |\nabla v(x)| \leq 1 & \text{for a.e. } x \in \mathbb{R}^n. \end{cases} \tag{4.6}$$

Finally, consider the extremal Pucci operators

$$M_\lambda^+ w(x) = \sup_{A\in\mathcal{M}_\lambda} L_A^s w(x) \quad \text{and} \quad M_\lambda^- w(x) = \inf_{A\in\mathcal{M}_\lambda} L_A^s w(x).$$

To prove Theorem 1.2(2), we need the following rescaled version of a regularity result from [2].

Proposition 4.1. *Let $\alpha \in (0, s)$, $1 + s + \alpha < 2$, $K > 0$, and $R \geq 1$. If w satisfies*

$$\begin{cases} w \geq 0 & \text{in } \mathbb{R}^n \\ D^2 w(x) \geq -K \operatorname{Id} & \text{for a.e. } x \in B_{2R} \\ M_\lambda^+(w - w(\cdot - h)) \geq -K|h| & \text{in } \{w>0\}\cap B_R \\ |\nabla w(x)| \leq K(1+|x|^{s+\alpha}) & \text{for a.e. } x \in \mathbb{R}^n, \end{cases}$$

then there exist $0 < \tau < 1$ and $C > 0$, depending on α and λ, such that

$$\|w\|_{L^\infty(B_{R/2})} + R\|\nabla w\|_{L^\infty(B_{R/2})} + R^{1+\tau}[\nabla w]_{C^\tau(B_{R/2})} \leq CKR^2.$$

Proof of Theorem 1.2(2). Fix $\mathcal{B}$ as in the statement. Let λ be as in (4.2), and let v be as in (1.8). Observe that, by (4.6) and (4.5),

$$\begin{aligned} M_\lambda^+(v - v(\cdot - h))(x) &\geq (L_A^s v + c_A)(x) - (L_A^s v + c_A)(x-h) - (c_A(x) - c_A(x-h)) \\ &\geq (L_A^s v + c_A)(x) - |h|, \end{aligned}$$

which gives

$$M_\lambda^+(v - v(\cdot - h)) \geq \sup_{A\in\mathcal{M}_\lambda} \left\{L_A^s v + c_A(x)\right\} - |h| = -|h| \quad \text{in } \{v>0\}\cap 2\mathcal{B}.$$

With this and (4.6), we can apply Proposition 4.1 and conclude that $v \in C^{1,\tau}(\mathcal{B})$, with the corresponding estimate. □

5. Proof of Theorem 1.3

In order to prove Theorem 1.3, we consider $v = \psi - u$ as in (1.8). We showed, in section 4, that v satisfies the locally uniformly elliptic obstacle problem (4.6) with ellipticity constants $\lambda > 0$ (as defined in (4.2)) and $1/\lambda^{(n-1)(n+2s)}$. Before proceeding with the proof, we define regular free boundary points, the constant $\bar{\alpha} > 0$, and the rescalings we will use to determine the blow up sequence, by following [2].

Definition 5.1. Let $\nu : (0,\infty) \to (0,\infty)$ be a nonincreasing function with

$$\lim_{r\searrow 0} \nu(r) = \infty.$$

We say that a free boundary point $x_0 \in \partial\{v > 0\}$ is *regular with modulus ν* if

$$\sup_{\rho\geq r} \frac{\sup_{B_\rho(x_0)} v}{\rho^{1+s+\alpha}} \geq \nu(r)$$

for some $\alpha \in (0, s)$ such that

$$1 + s + \alpha < 2.$$

Definition 5.2. Let $\bar{\alpha} = \bar{\alpha}(n, s, \lambda) > 0$ be the minimum of the following three constants:
- The $\alpha > 0$ of the interior C^α estimate given by [5, Theorem 11.1];
- The $\alpha > 0$ of the boundary C^α estimate for u/d^s given by [11, Proposition 1.1];
- The $\alpha > 0$ of the interior $C^{2s+\alpha}$ estimate for convex equations given by [4, Theorem 1.1] and [13, Theorem 1.1].

Without loss of generality and for the rest of this section, we assume that $x_0 = 0$ is a regular free boundary point with modulus ν. In the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, we further assume that

$$\liminf_{\rho \searrow 0} \frac{|\{v = 0\} \cap B_\rho|}{|B_\rho|} > 0.$$

In particular, there exists $c_0 > 0$ such that

$$\frac{|\{v = 0\} \cap B_\rho|}{|B_\rho|} \geq c_0 > 0 \quad \text{for all } \rho \text{ sufficiently small.} \tag{5.1}$$

For $r > 0$, we define the rescalings

$$v_r(x) = \frac{v(rx)}{r^{1+s+\alpha}\theta(r)} \quad \text{for } x \in \mathbb{R}^n \tag{5.2}$$

where

$$\theta(r) = \sup_{\rho \geq r} \frac{\|\nabla v\|_{L^\infty(B_\rho)}}{\rho^{s+\alpha}}.$$

Then, θ is nonincreasing and $\theta(r) \geq \nu(r)$ for all $r > 0$, see [2, Lemma 5.4].

Proof of Theorem 1.3. We first prove that for any $R > 0$, $\|v_r\|_{C^{1,\tau}(B_R)}$ is uniformly bounded for all $r > 0$ sufficiently small, where $\tau \in (0, 1)$ is as in Proposition 4.1. Observe that

$$v_r \geq 0 \quad \text{in } \mathbb{R}^n, \tag{5.3}$$

and, since $1 + s + \alpha < 2$ and $\theta(r) \geq \nu(r)$, for all $r < 1$,

$$D^2 v_r(x) = \frac{r^2}{r^{1+s+\alpha}\theta(r)} D^2 v(rx) \geq -\frac{1}{\nu(r)} \operatorname{Id} \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{5.4}$$

Let $c_A(x)$ be as in (4.4) and define

$$c_{A,r}(x) = \frac{c_A(rx)}{r^{1-s+\alpha}\theta(r)}.$$

Then, for every $r < 1$, since $\alpha \in (0, s)$ and (4.5) holds,

$$[c_{A,r}]_{\mathrm{Lip}(\mathbb{R}^n)} \leq \frac{r^{s-\alpha}}{\theta(r)} [c_A]_{\mathrm{Lip}(\mathbb{R}^n)} \leq \frac{1}{\nu(r)}.$$

Hence, following the proof of Theorem 1.2(2) in section 4, we see that

$$M_\lambda^+(v_r - v_r(\cdot - h))(x) \geq -\frac{|h|}{\nu(r)} \quad \text{for all } x \in \{v_r > 0\} \cap B_{4R} \tag{5.5}$$

and for all $r < 1$ sufficiently small. Finally,

$$\|\nabla v_r\|_{L^\infty(B_R)} = \frac{R^{s+\alpha}\|\nabla v\|_{L^\infty(B_{Rr})}}{(Rr)^{s+\alpha}\theta(r)} \leq R^{s+\alpha}, \tag{5.6}$$

which implies that

$$|\nabla v_r(x)| \leq 2(1 + |x|^{s+\alpha}) \quad \text{for all } x \in \mathbb{R}^n.$$

Thus, by Proposition 4.1, $\|v_r\|_{C^{1,\tau}(B_R)}$ is uniformly bounded for all $r > 0$ sufficiently small.

Next, for any $k \geq 1$, we choose

$$r_k \geq \frac{1}{k} \tag{5.7}$$

such that

$$\frac{\|\nabla v\|_{L^\infty(B_{r_k})}}{r_k^{s+\alpha}} \geq \frac{1}{2}\theta(1/k) \geq \frac{1}{2}\theta(r_k). \tag{5.8}$$

Since $\|\nabla v\|_{L^\infty(\mathbb{R}^n)} \leq 1$ and $\theta(1/k) \geq \nu(1/k) \to \infty$ as $k \to \infty$, we have that $r_k \to 0$ as $k \to \infty$. In addition, from (5.8) it follows that

$$\|\nabla v_{r_k}\|_{L^\infty(B_1)} \geq \frac{1}{2}. \tag{5.9}$$

Moreover, in the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, by (5.1),

$$\frac{|\{v_{r_k} = 0\} \cap B_\rho|}{|B_\rho|} = \frac{|\{v = 0\} \cap B_{r_k\rho}|}{|B_{r_k\rho}|} \geq c_0 > 0 \quad \text{for all } \rho \text{ sufficiently small.} \tag{5.10}$$

By Arzelà–Ascoli's theorem and a standard diagonal argument, there exists $v_0 \in C^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$v_{r_k} \to v_0 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Additionally, by (5.3) and (5.4),

$$\begin{cases} v_0 \geq 0 & \text{in } \mathbb{R}^n \\ D^2 v_0(x) \geq 0 & \text{for a.e. } x \in \mathbb{R}^n, \end{cases}$$

and, by (5.6) and (5.9),

$$\frac{1}{2} \leq \|\nabla v_0\|_{L^\infty(B_R)} \leq R^{s+\alpha} \quad \text{for all } R \geq 1.$$

Also, in the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, from (5.10), we find that

$$\liminf_{\rho \searrow 0} \frac{|\{v_0 = 0\} \cap B_\rho|}{|B_\rho|} \geq c_0 > 0.$$

Next, as $R > 0$ in (5.5) was arbitrary, by passing to the limit as $|h| \to 0$ and $r = r_k \to 0$, we get

$$M^+_\lambda(\partial_{\mathrm{e}} v_0) \geq 0 \quad \text{in } \{v_0 > 0\}$$

and for all $\mathrm{e} \in \mathbb{S}^{n-1}$. Furthermore, by arguing as we did to obtain (5.5), for any $R > 0$ and any nonnegative probability measure μ with compact support, we find that

$$M^+_\lambda\left(v_r - \int v_r(\cdot - h)\, d\mu(h)\right) \geq -\frac{|h|}{\nu(r)} \quad \text{in } \{v_r > 0\} \cap B_R$$

provided r is sufficiently small. As a consequence,

$$M^+_\lambda\left(v_0 - \int v_0(\cdot - h)\, d\mu(h)\right) \geq 0 \quad \text{in } \{v_0 > 0\}.$$

Therefore, applying the classification results in [2, Theorems 7.1, 7.2] to v_0, we finally obtain

$$v_0(x) = K_0(\mathrm{e}_0 \cdot x)_+^{1+s},$$

for some $1/4 \leq K_0 \leq 1$ and $\mathrm{e}_0 \in \mathbb{S}^{n-1}$, as desired. □

6. Proof of Theorem 1.4

As in section 5, we assume, without loss of generality, that $x_0 = 0$ is a regular free boundary point with modulus ν.

Proof of Theorem 1.4. Consider v_r as in (5.2), for $r = r_k$ given by (5.7). From Theorem 1.3 and its proof, we have that given any $\delta_0 > 0$ and $R_0 \geq 1$, there exists $r_0 = r_0(\delta_0, R_0, \alpha, \nu, \lambda) \in (0,1)$ such that for all $0 < r_k < r_0$,

$$\begin{cases} M_\lambda^+(\partial_{\mathrm{e}} v_{r_k}) \geq -\delta_0 & \text{in } \{v_{r_k} > 0\} \cap B_{R_0} \\ M_\lambda^-(\partial_{\mathrm{e}} v_{r_k}) \leq \delta_0 & \text{in } \{v_{r_k} > 0\} \cap B_{R_0}, \end{cases}$$

for all $\mathrm{e} \in \mathbb{S}^{n-1}$, and

$$|v_{r_k}(x) - K_0(\mathrm{e}_0 \cdot x)_+^{1+s}| + |\nabla v_{r_k}(x) - (1+s)K_0(\mathrm{e}_0 \cdot x)_+^s \mathrm{e}_0| < \delta_0 \quad \text{for every } x \in B_{R_0}.$$

In addition, for every $R \geq 1$, we have

$$\|\nabla v_{r_k}\|_{L^\infty(B_R)} \leq R^{s+\alpha},$$

for all $k \geq 1$, see (5.6). With these estimates in hand, we can argue exactly as in sections 8.2 and 8.3 of [2] and deduce that Theorem 1.4 holds. $\square$

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