

# HIGHER REGULARITY OF THE FREE BOUNDARY IN THE OBSTACLE PROBLEM FOR THE FRACTIONAL LAPLACIAN

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ABSTRACT. We prove a higher regularity result for the free boundary in the obstacle problem for the fractional Laplacian via a higher order boundary Harnack estimate.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we investigate the higher regularity of the free boundary in the fractional obstacle problem. We prove a higher order boundary Harnack estimate, building on ideas developed by De Silva and Savin in [12, 13, 14]. As a consequence, we show that if the obstacle is  $C^{m,\beta}$ , then the free boundary is  $C^{m-1,\alpha}$  near regular points for some  $0 < \alpha \leq \beta$ . In particular, smooth obstacles yield smooth free boundaries near regular points.

**1.1. The Fractional Obstacle Problem.** For a given function (obstacle)  $\varphi \in C(\mathbb{R}^n)$  decaying rapidly at infinity and  $s \in (0, 1)$ , a function  $v$  is a solution of the fractional obstacle problem if

$$\begin{cases} v(x) \geq \varphi(x) & \text{in } \mathbb{R}^n \\ \lim_{|x| \rightarrow \infty} v(x) = 0 & \text{on } \mathbb{R}^n \\ (-\Delta)^s v(x) \geq 0 & \text{in } \mathbb{R}^n \\ (-\Delta)^s v(x) = 0 & \text{in } \{v > \varphi\} \end{cases} \quad (1.1)$$

where the  $s$ -Laplacian  $(-\Delta)^s$  of a function  $u$  is defined by

$$(-\Delta)^s u(x) := c_{n,s} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz.$$

The sets

$$\mathcal{P} := \{v = \varphi\} \quad \text{and} \quad \Gamma := \partial\{v = \varphi\}$$

are known as the *contact set* and the *free boundary* respectively.

The fractional obstacle problem appears in many contexts, including the pricing of American options with jump processes (see [11] and the Appendix of [3] for an informal discussion) and the study of the regularity of minimizers of nonlocal interaction energies in kinetic equations (see [10]).

While the obstacle problem for the fractional Laplacian is nonlocal, it admits a local formulation thanks to the extension method (see [9, 24]). Specifically, one considers the  $a$ -harmonic<sup>1</sup> extension  $\tilde{v}$  of  $v$  to the upper half-space  $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times (0, \infty)$ :

$$\begin{cases} L_a \tilde{v}(x, y) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ \tilde{v}(x, 0) = v(x) & \text{on } \mathbb{R}^n \end{cases}$$

where

$$L_a u(x, y) := \text{div}(|y|^a \nabla u(x, y)) \quad \text{and} \quad a := 1 - 2s \in (-1, 1).$$

The function  $\tilde{v}$  is obtained as the minimizer of the variational problem

$$\min \left\{ \int_{\mathbb{R}_+^{n+1}} |\nabla u|^2 |y|^a dx dy : u \in H^1(\mathbb{R}_+^{n+1}, |y|^a), u(x, 0) = v(x) \right\}$$

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<sup>1</sup>We say a function  $u$  is  $a$ -harmonic if  $L_a u = 0$ .

and satisfies

$$-\lim_{y \rightarrow 0} |y|^a \tilde{v}(x, y) = (-\Delta)^s v(x) \quad \forall x \in \mathbb{R}^n.$$

After an even reflection across the hyperplane  $\{y = 0\}$ , (1.1) is equivalent to the following local problem. For a given  $a \in (-1, 1)$  and a function  $\varphi \in C(\mathbb{R}^n)$  decaying rapidly at infinity, a function  $\tilde{v}$  is a solution of the localized fractional obstacle problem if it is even in  $y$  and satisfies

$$\begin{cases} \tilde{v}(x, 0) \geq \varphi(x) & \text{on } \mathbb{R}^n \\ \lim_{|(x,y)| \rightarrow \infty} \tilde{v}(x, y) = 0 & \text{on } \mathbb{R}^{n+1} \\ L_a \tilde{v}(x, y) \leq 0 & \text{in } \mathbb{R}^{n+1} \\ L_a \tilde{v}(x, y) = 0 & \text{in } \mathbb{R}^{n+1} \setminus \{\tilde{v}(x, 0) = \varphi(x)\}. \end{cases} \quad (1.2)$$

When  $a = 0$ , i.e.,  $s = 1/2$ , the operator  $L_a$  is the Laplacian, and (1.2) is the well-known Signorini (thin obstacle) problem, which can be stated not only in all of  $\mathbb{R}^{n+1}$ , but in suitable bounded domains of  $\mathbb{R}^{n+1}$ . For example, a typical formulation of the Signorini problem is in  $B_1 \subset \mathbb{R}^{n+1}$ :

$$\begin{cases} \tilde{v}(x, 0) \geq \varphi(x) & \text{on } B_1 \cap \{y = 0\} \\ \tilde{v}(x, y) = g(x, y) & \text{on } \partial B_1 \\ \Delta \tilde{v}(x, y) \leq 0 & \text{in } B_1 \\ \Delta \tilde{v}(x, y) = 0 & \text{in } B_1 \setminus \{\tilde{v}(x, 0) = \varphi(x)\}. \end{cases} \quad (1.3)$$

Primary questions in obstacle problems are the regularity of the solution and the structure and regularity of the free boundary. The local formulation of the fractional obstacle problem, (1.2), allows the use of local PDE techniques to study the regularity of the solution and the free boundary. Under mild conditions on the obstacle<sup>2</sup>, Caffarelli, Salsa, and Silvestre show, in [8], that the solution of (1.1) is optimally  $C^{1,s}$  using the almost-optimal regularity of the solution shown via potential theoretic techniques in [28]. Furthermore, studying limits of appropriate rescalings of the solution (blowups) at points on the free boundary, they show the blowup at  $x_0 \in \Gamma$  must either have (a) homogeneity  $1 + s$  or (b) homogeneity at least 2. The points at which the blowup is  $(1 + s)$ -homogeneous are known as *regular points* of  $\Gamma$ . In [8], they show that the regular points form a relatively open subset of  $\Gamma$  and that  $\Gamma$  is  $C^{1,\sigma}$  near these regular points for some  $0 < \sigma < 1$ . For the case  $s = 1/2$ , analogous results were first shown in [1, 2]. The structure of the free boundary away from regular points is investigated, for example, in [3] and [18].

**1.2. Main Result and Current Literature.** Our main result is the following:

**Theorem 1.1.** *Let  $\varphi \in C^{m,\beta}(\mathbb{R}^n)$  with  $m \geq 4$  and  $\beta \in (0, 1)$  or  $m = 3$  and  $\beta = 1$ . Suppose  $x_0$  is a regular point of the free boundary  $\Gamma = \Gamma(v)$  of the solution  $v$  to (1.1). Then,  $\Gamma \in C^{m-1,\alpha}$  in a neighborhood of  $x_0$  for some  $\alpha \in (0, 1)$  depending on  $s, n, m$ , and  $\beta$ . In particular, if  $\varphi \in C^\infty(\mathbb{R}^n)$ , then  $\Gamma \in C^\infty$  near regular points.*

Starting from the  $C^{1,\sigma}$  regularity obtained in [8], the Hölder exponent  $\alpha$  obtained in Theorem 1.1 is the minimum of  $\beta$  and  $\sigma$ . In order to prove Theorem 1.1, we establish a higher order boundary Harnack estimate for the operator  $L_a$ . We prove this estimate in slit domains, that is, domains in  $\mathbb{R}^{n+1}$  from which an  $n$ -dimensional slit  $\mathcal{P} \subset \{x_{n+1} = 0\}$  is removed. Recall that the classical boundary Harnack principle states that the quotient of two positive harmonic functions that vanish continuously on a portion of the boundary of a Lipschitz domain is Hölder continuous up to the boundary (see [7, 17]). In [13], De Silva and Savin remarkably extend this idea to the *higher order boundary Harnack principle*, proving that the quotient of two positive harmonic functions that vanish continuously on a portion of the boundary of a  $C^{k,\alpha}$  domain is  $C^{k,\alpha}$ . Motivated by applications to the Signorini problem, in [14], they prove such a higher order boundary Harnack principle on slit domains. To do so, they assume  $\Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$  is locally the graph of a function of the first  $n - 1$  variables, and they consider a coordinate system  $(x, r)$  on  $\mathbb{R}^{n+1}$  where  $x \in \mathbb{R}^n$  and  $r$  is the distance to  $\Gamma$ . They also define a

<sup>2</sup>In [8], the obstacle is assumed to be  $C^{2,1}$ , though in [6], this is relaxed, and  $\varphi$  is only assumed to be  $C^{1,s+\varepsilon}$ .

corresponding notion of Hölder regularity  $C_{x,r}^{k,\alpha}$  (see Section 2) that restricts to the standard notion of  $C^{k,\alpha}$  on  $\Gamma$  when  $\Gamma \in C^{k,\alpha}$ .

**Theorem 1.2** (De Silva and Savin, [14]). *Let  $u$  and  $U > 0$  be harmonic functions in  $B_1 \setminus \mathcal{P} \subset \mathbb{R}^{n+1}$  that are even in  $x_{n+1}$  and vanish continuously on the slit  $\mathcal{P}$ . Suppose  $0 \in \Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$  with  $\Gamma \in C^{k,\alpha}$  for  $k \geq 1$  and  $\|\Gamma\|_{C^{k,\alpha}} \leq 1$ . If  $\|u\|_{L^\infty(B_1)} \leq 1$  and  $U(\nu(0)/2) = 1$ , where  $\nu$  is the outer unit normal to  $\mathcal{P}$ , then*

$$\left\| \frac{u}{U} \right\|_{C_{x,r}^{k,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

for some  $C = C(n, k, \alpha) > 0$ .

Here,  $\|\Gamma\|_{C^{k,\alpha}}$  is defined as in (2.1). Theorem 1.2 is used to prove that the free boundary is smooth near regular points for (1.3) with  $\varphi \equiv 0$  in the same way that the boundary Harnack principle is used to improve the regularity of the free boundary near regular points from Lipschitz to  $C^{1,\alpha}$  in, for example, the classical obstacle problem. Indeed, if  $\Gamma$  is locally the graph of a  $C^{k,\alpha}$  function  $\gamma$  of the first  $n - 1$  variables, then Theorem 1.2 implies that  $\partial_i \tilde{v} / \partial_n \tilde{v}$  is  $C^{k,\alpha}$  on  $\Gamma$  where  $\tilde{v}$  is the solution to the Signorini problem. On the other hand,  $\Gamma$  is also the zero level set of  $\tilde{v}$ , and so  $\partial_i \gamma$ , for each  $i = 1, \dots, n - 1$ , is given by  $\partial_i \tilde{v} / \partial_n \tilde{v}$  on  $\Gamma$ . Hence,  $\gamma \in C^{k+1,\alpha}$ . Starting with  $k = 1$  and proceeding iteratively, one proves that the free boundary is smooth near regular points.

The proof of the higher order boundary Harnack estimate in this paper is motivated by the global strategy developed by De Silva and Savin to prove Theorem 1.2. However, some delicate arguments are needed to adapt these ideas to our setting, which we briefly describe here. The proof involves a perturbative argument, the core regularity result being one in which  $\Gamma$  is flat (Proposition 3.3). When  $a = 0$ , the flat case follows from capitalizing on the structural symmetry of the Laplacian; boundary regularity is inherited from interior regularity for a reflection of the solution. Instead, to handle the case  $a \neq 0$ , we prove the necessary regularity of our solutions by hand. First, a reduction argument allows us to focus on the two-dimensional case. Here, the specific degeneracy of the operator  $L_a$ , for each  $a \neq 0$ , forces a specific degeneracy in solutions that vanish on the negative  $x$ -axis to the equation  $L_a u = 0$ . We observe this first in global homogeneous solutions. Then, we prove that our solutions have a well-defined power series-like decomposition in terms of these homogeneous solutions, which in turn yields the regularity result. (See Section 6.)

Another new difficulty we encounter comes from considering nonzero obstacles in (1.1). As discussed above, Theorem 1.2 implies  $C^\infty$  regularity of the free boundary near regular points in the Signorini problem with zero obstacle, that is, taking  $\varphi \equiv 0$  in (1.3). Yet, taking  $\varphi \equiv 0$  in the nonlocal problem (1.1) is rather uninteresting: the solution is identically zero. While the results of [14] do extend to (1.1) when  $s = 1/2$  so long as an extension of  $\varphi$  can be subtracted off without producing a right-hand side<sup>3</sup>, such an extension is not generally possible. The new feature of the higher order boundary Harnack estimates here, necessary for treating general obstacles, is that we allow both  $L_a u$  and  $L_a U$  to be nonzero when considering the quotient  $u/U$ . Still, handling general obstacles in (1.1) is quite involved. Even after constructing a suitable extension of the obstacle from  $\mathbb{R}^n$  to  $\mathbb{R}^{n+1}$ , one finds a large gap between having and applying these propositions, another consequence of having to work with a degenerate elliptic operator. (See Section 7.)

Proposition 3.2 and Proposition 5.4 are the higher order boundary Harnack estimates in the generality needed to prove Theorem 1.1. The simplest case of these estimates, however, takes the following form.

**Theorem 1.3** (Higher Order Boundary Harnack Estimate). *Let  $u$  and  $U > 0$  be  $a$ -harmonic functions in  $B_1 \setminus \mathcal{P} \subset \mathbb{R}^{n+1}$  that are even in  $x_{n+1}$  and vanish continuously on the slit  $\mathcal{P}$ . Suppose  $0 \in \Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$  with  $\Gamma \in C^{k,\alpha}$  for  $k \geq 1$  and  $\|\Gamma\|_{C^{k,\alpha}} \leq 1$ . If  $\|u\|_{L^\infty(B_1)} \leq 1$  and  $U(\nu(0)/2) = 1$ , where  $\nu$  is the outer*

<sup>3</sup>For instance, this can be done if  $\Delta^m \varphi = 0$  for some  $m \in \mathbb{N}$ .

unit normal to  $\mathcal{P}$ , then

$$\left\| \frac{u}{U} \right\|_{C_{x,r}^{k,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

for some  $C = C(a, n, k, \alpha) > 0$ .

The original approach to proving higher regularity in obstacle problems, pioneered in [20], was to use the hodograph-Legendre transform. These techniques have been extended to prove higher regularity in the Signorini problem with zero obstacle in [21] and in thin obstacle problems with variable coefficients and inhomogeneities in [22]. On the other hand, at the same time as [21], De Silva and Savin used the higher order boundary Harnack principle to show higher free boundary regularity in the Signorini problem with zero obstacle in [14], as we discussed above. They also used these techniques to give a new proof of higher regularity of the free boundary in the classical obstacle problem (see [13]). The higher order boundary Harnack approach has been adapted to the parabolic setting, proving higher regularity of the free boundary for the parabolic obstacle problem in [4] and for the parabolic Signorini problem with zero obstacle in [5]. We mention that an advantage of the hodograph-Legendre transform approach is that it allows one to prove analyticity of the free boundary near regular points when the obstacle is analytic.

Upon completion of this work, we learned that Koch, Rüländ, and Shi in [23] – independently and at the same time – have proven an analogous result to our Theorem 1.1. In contrast to the methods used herein, they employ the partial hodograph-Legendre transform techniques mentioned above.

**1.3. Organization.** In Section 2, we introduce some notation and present some useful properties of the operator  $L_a$ . In Section 3, we state and prove a pointwise higher order boundary Harnack estimate. Section 4 is dedicated to proving a pointwise Schauder estimate, while Section 5 extends the results of the previous two sections to a lower regularity setting. In Section 6, we prove a regularity result for  $a$ -harmonic functions vanishing continuously on a hyperplane. Finally, we prove Theorem 1.1 in Section 7.

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## 2. PRELIMINARIES

**2.1. Notation and Terminology.** Let  $X \in \mathbb{R}^{n+1}$  be given by

$$X = (x', x_n, y) = (x', z) = (x, y)$$

where  $x' \in \mathbb{R}^{n-1}$ ,  $x_n \in \mathbb{R}$  is the  $n$ th component of  $X$ , and  $y \in \mathbb{R}$  is the  $(n+1)$ st component of  $X$ . In this way,  $x = (x', x_n) \in \mathbb{R}^n$  and  $z = (x_n, y) \in \mathbb{R}^2$ . Furthermore, define

$$B_\lambda := \{|X| < \lambda\}, \quad B_\lambda^* := \{|x| < \lambda\}, \quad \text{and} \quad B'_\lambda := \{|x'| < \lambda\}.$$

Let  $\Gamma$  be a codimension two surface in  $\mathbb{R}^{n+1}$  of class  $C^{k+2,\alpha}$  with  $k \geq -1$ . Then, up to translation, rotation, and dilation,  $\Gamma$  is locally given by

$$\Gamma = \{(x', \gamma(x'), 0) : x' \in B'_1\}$$

where  $\gamma : B'_1 \rightarrow \mathbb{R}$  is such that

$$\gamma(0) = 0, \quad \nabla_{x'} \gamma(0) = 0, \quad \text{and} \quad \|\gamma\|_{C^{k+2,\alpha}(B'_1)} \leq 1.$$

We let

$$\|\Gamma\|_{C^{k,\alpha}} := \|\gamma\|_{C^{k,\alpha}(B'_1)}. \tag{2.1}$$

Define the  $n$ -dimensional *slit*  $\mathcal{P}$  by

$$\mathcal{P} := \{x_n \leq \gamma(x'), y = 0\},$$

and notice that  $\partial_{\mathbb{R}^n} \mathcal{P} = \Gamma$ .

Let  $d = d(x)$  denote the signed distance in  $\mathbb{R}^n$  from  $x$  to  $\Gamma$  with  $d > 0$  in the  $e_n$ -direction, and let  $r = r(X)$  be the distance to  $\Gamma$  in  $\mathbb{R}^{n+1}$ :

$$r := (y^2 + d^2)^{1/2}.$$

Then,

$$\nabla_x r = \frac{d}{r} \nu, \quad \nabla_x d = \nu, \quad \text{and} \quad -\Delta_x d = \kappa$$

where  $\kappa = \kappa(x)$  denotes the mean curvature and  $\nu = \nu(x)$  represents the unit normal in  $\mathbb{R}^n$  of the parallel surface to  $\Gamma$  passing through  $x$ . Moreover, set

$$U_a := \left( \frac{r+d}{2} \right)^s. \quad (2.2)$$

Observe that one can express  $U_a$  as

$$U_a = \frac{|y|^{2s}}{2^s (r-d)^s}, \quad (2.3)$$

and when  $\Gamma$  is flat, that is,  $\gamma \equiv 0$ ,  $U_a$  is equal to

$$\bar{U}_a := \left( \frac{|z|+d}{2} \right)^s. \quad (2.4)$$

As shown in [8],  $\bar{U}_a$  is (up to multiplication by a constant) the only positive  $a$ -harmonic function vanishing on  $\{x_n \leq 0, y = 0\}$ . Thus, if  $\tilde{v}$  is a global homogeneous solution of (1.2) with  $\varphi \equiv 0$ , then, up to a rotation,  $\bar{U}_a = \partial_\nu \tilde{v}$ , where  $\nu$  is the unit normal in  $\mathbb{R}^n$  to the free boundary  $\Gamma$ . When  $\Gamma$  is not flat, the function  $U_a$  is not  $a$ -harmonic. However, it approximates  $a$ -harmonic functions in the sense of the Schauder estimates of Proposition 4.1.

We work in the coordinate system determined by  $x$  and  $r$ . For a polynomial

$$P = P(x, r) = p_{\mu m} x^\mu r^m,$$

we let

$$\|P\| := \max\{|p_{\mu m}|\}.$$

Here,  $\mu$  is a multi-index,  $|\mu| = \mu_1 + \dots + \mu_n$ ,  $\mu_i \geq 0$ , and  $x^\mu = x_1^{\mu_1} \dots x_n^{\mu_n}$ . It is useful to think that the coefficients  $p_{\mu m}$  are defined for all indices  $(\mu, m)$  by extending by zero. Frequently, we use the convention of summation over repeated indices.

A function  $f : B_1 \rightarrow \mathbb{R}$  is *pointwise*  $C_{x,r}^{k,\alpha}$  at  $X_0 \in \Gamma$  if there exists a (tangent) polynomial  $P(x, r)$  of degree  $k$  such that

$$f(X) = P(x, r) + O(|X - X_0|^{k+\alpha}).$$

We will write  $f \in C_{x,r}^{k,\alpha}(X_0)$ , and  $\|f\|_{C_{x,r}^{k,\alpha}(X_0)}$  will denote the smallest constant  $M > 0$  for which

$$\|P\| \leq M \quad \text{and} \quad |f(X) - P(x, r)| \leq M|X - X_0|^{k+\alpha}.$$

We will call objects *universal* if they depend only on any or all of  $a, n, k$ , or  $\alpha$ . Throughout, unless otherwise stated,  $C$  and  $c$  will denote positive universal constants that may change from line to line.

**2.2. Basic Regularity Results for  $L_a$ .** Let us collect some regularity results for weak solutions  $u \in H^1(B_\lambda, |y|^\alpha)$  of the equation  $L_a u = |y|^\alpha f$ , beginning with interior regularity. Throughout the section, we assume that  $0 < \lambda \leq 1$ .

As the weight  $|y|^\alpha$  is an  $A_2$ -Muckenhoupt weight, we obtain the following local boundedness property for subsolutions  $L_a$  from [16].

**Proposition 2.1** (Local Boundedness). *Let  $u \in H^1(B_\lambda, |y|^\alpha)$  be such that  $L_a u \geq |y|^\alpha f$  in  $B_\lambda$  with  $f \in L^\infty(B_\lambda)$ . Then,*

$$\sup_{B_{\lambda/2}} u \leq C \left( \frac{1}{\lambda^{n+1+a}} \int_{B_\lambda} |u|^2 |y|^\alpha dX \right)^{1/2} + C\lambda^2 \|f\|_{L^\infty(B_\lambda)}. \quad (2.5)$$

*Proof.* The inequality follows from [16, Theorem 2.3.1] applied to  $u = c(\lambda^2 - |X|^2)$  and  $u + c|X|^2$ ; see [3, Lemma 3.4] for details.<sup>4</sup>  $\square$

The operator  $L_a$  also enjoys a Harnack inequality [16, Lemma 2.3.5] and a boundary Harnack inequality [15, p. 585]. By a standard argument (see, for instance, [19, Theorem 8.22]), the Harnack inequality implies that solutions of  $L_a u = |y|^a f$  for bounded  $f$  are Hölder continuous.

**Proposition 2.2** (Hölder Continuity). *Let  $u \in H^1(B_\lambda, |y|^a)$  be such that  $L_a u = |y|^a f$  in  $B_\lambda$  with  $f \in L^\infty(B_\lambda)$ . Then,*

$$\|u\|_{C^{0,\alpha}(B_{\lambda/2})} \leq C\lambda^{-\alpha}\|u\|_{L^\infty(B_\lambda)} + C\lambda^{2-\alpha}\|f\|_{L^\infty(B_\lambda)}$$

for some  $\alpha \in (0, 1)$ .

If  $u$  is such that  $L_a u = 0$ , then  $L_a(\partial_i u) = 0$  for  $i = 1, \dots, n$ . Moreover, as pointed out in [9], the function  $|y|^a \partial_y u$  satisfies  $L_{-a}(|y|^a \partial_y u) = 0$ . If instead  $L_a u = |y|^a f$  for  $f$  bounded, then one can show that

$$L_a(\partial_i u) = |y|^a \partial_i f \quad \forall i = 1, \dots, n \quad \text{and} \quad L_{-a}(|y|^a \partial_y u) = \partial_y f.$$

Here, the partial derivatives of  $f$  are understood in the distributional sense. And so, we have the following regularity result for  $\nabla_x u$  and  $|y|^a \partial_y u$ .

**Proposition 2.3** (Interior Regularity of  $\nabla_x u$  and  $|y|^a \partial_y u$ ). *Let  $u \in H^1(B_\lambda, |y|^a)$  be such that  $L_a u = |y|^a f$  in  $B_\lambda$  with  $f \in L^\infty(B_\lambda)$ . Then,*

$$\|\nabla_x u\|_{L^\infty(B_{\lambda/4})} \leq C\lambda^{-1}\|u\|_{L^\infty(B_\lambda)} + C\|f\|_{L^\infty(B_\lambda)}.$$

Furthermore, if  $g := |y|^a \partial_y f \in L^\infty(B_\lambda)$ , then

$$\||y|^a \partial_y u\|_{L^\infty(B_{\lambda/4})} \leq C\lambda^{a-1}\|u\|_{L^\infty(B_\lambda)} + C\lambda^a\|f\|_{L^\infty(B_\lambda)} + C\lambda^2\|g\|_{L^\infty(B_\lambda)}.$$

In fact,  $\partial_i u$  and  $|y|^a \partial_y u$  are Hölder continuous, but boundedness is all we need.

*Proof.* By [16, Theorems 2.3.1 and 2.3.14],  $\nabla_x u$  has the following local boundedness property:

$$\sup_{B_{\lambda/4}} |\nabla_x u| \leq C \left( \frac{1}{\lambda^{n+1+a}} \int_{B_{\lambda/2}} |\nabla_x u|^2 |y|^a dX \right)^{1/2} + C\lambda^\delta \|f\|_{L^\infty(B_{\lambda/2})}$$

for some  $\delta > 0$ . Using the energy inequality

$$\int_{B_{\lambda/2}} |\nabla u|^2 |y|^a dX \leq C \int_{B_\lambda} |f|^2 |y|^a dX + \frac{C}{\lambda^2} \int_{B_\lambda} |u|^2 |y|^a dX \quad (2.6)$$

for  $u$  and recalling  $\lambda \leq 1$ , the first estimate follows.

Let  $w := |y|^a \partial_y u$  and note that  $L_{-a} w = |y|^{-a} g$ . Since  $g \in L^\infty(B_\lambda)$ , (2.5) implies that

$$\begin{aligned} \|w\|_{L^\infty(B_{\lambda/4})} &\leq C \left( \frac{1}{\lambda^{n+1-a}} \int_{B_{\lambda/2}} |w|^2 |y|^{-a} dX \right)^{1/2} + C\lambda^2 \|g\|_{L^\infty(B_{\lambda/2})} \\ &\leq C \left( \frac{1}{\lambda^{n+1-a}} \int_{B_{\lambda/2}} |\nabla u|^2 |y|^a dX \right)^{1/2} + C\lambda^2 \|g\|_{L^\infty(B_{\lambda/2})}. \end{aligned}$$

Applying (2.6) once more concludes the proof.  $\square$

The following boundary regularity result is a consequence of Proposition 2.3 applied to odd reflections. More specifically, let  $B_\lambda^+ := B_\lambda \cap \{y > 0\}$  and  $u \in H^1(B_\lambda^+, y^a)$  be such that

$$\begin{cases} L_a u = y^a f & \text{in } B_\lambda^+ \\ u = 0 & \text{on } \{y = 0\}, \end{cases} \quad (2.7)$$

<sup>4</sup>We caution the reader that the authors in [3] define  $L_a$  with the opposite sign convention, that is, they consider  $L_a u(x, y) := -\operatorname{div}(|y|^a \nabla u(x, y))$ .

and let  $\bar{u}$  and  $\bar{f}$  be the odd extensions across  $\{y = 0\}$  of  $u$  and  $f$  respectively. Then, notice that  $L_a \bar{u} = |y|^a \bar{f}$  in  $B_\lambda$ . Applying Proposition 2.3 to  $\bar{u}$  yields the following.

**Corollary 2.4** (Boundary Regularity for  $\nabla_x u$  and  $y^a \partial_y u$ ). *Suppose  $u \in H^1(B_\lambda^+, y^a)$  satisfies (2.7) where  $f \in C(\bar{B}_\lambda^+)$ . Then,*

$$\|\nabla_x u\|_{L^\infty(B_{\lambda/4}^+)} \leq C\lambda^{-1}\|u\|_{L^\infty(B_\lambda^+)} + C\|f\|_{L^\infty(B_\lambda^+)}.$$

Furthermore, if  $f$  vanishes on  $\{y = 0\}$  and  $y^a \partial_y f \in L^\infty(B_\lambda^+)$ , then letting  $g := y^a \partial_y f$ ,

$$\|y^a \partial_y u\|_{L^\infty(B_{\lambda/4}^+)} \leq C\lambda^{a-1}\|u\|_{L^\infty(B_\lambda^+)} + C\lambda^a\|f\|_{L^\infty(B_\lambda^+)} + C\lambda^2\|g\|_{L^\infty(B_\lambda^+)}.$$

If  $f$  does not vanish on  $\{y = 0\}$  yet depends only on  $x$ , then we have the following.

**Corollary 2.5** (Boundary Regularity of  $y^a \partial_y u$ ). *Suppose  $u \in H^1(B_\lambda^+, y^a)$  satisfies (2.7) where  $f = f(x)$  and  $f \in L^\infty(B_\lambda^+)$ . Then,*

$$\|y^a \partial_y u\|_{L^\infty(B_{\lambda/4}^+)} \leq C\lambda^{a-1}\|u\|_{L^\infty(B_\lambda^+)} + C\lambda^a\|f\|_{L^\infty(B_\lambda^+)}.$$

*Proof.* Letting  $w := |y|^a \partial_y \bar{u}$ , where again  $\bar{u}$  is the odd extension of  $u$  across  $\{y = 0\}$ , we have

$$L_{-a} w = 2f \mathcal{H}^n \llcorner \{y = 0\}.$$

Let  $M := \|f\|_{L^\infty(B_\lambda^+)}$  and consider the barriers

$$v_\pm := w \pm \frac{M}{1+a} |y|^{a+1},$$

which satisfy

$$L_{-a} v_\pm = (2f \pm 2M) \mathcal{H}^n \llcorner \{y = 0\}.$$

Therefore,  $L_{-a} v_+ \geq 0$  and  $L_{-a} v_- \leq 0$ . Applying Proposition 2.1 and arguing as in the proof of Proposition 2.3, we see that

$$\sup_{B_{\lambda/4}} v_+ \leq C\lambda^{a-1}\|u\|_{L^\infty(B_\lambda)} + C\lambda^a\|f\|_{L^\infty(B_\lambda)}.$$

The same bound holds for  $\sup_{B_{\lambda/4}} -v_-$ . As  $v_- \leq w \leq v_+$ , this concludes the proof.  $\square$

As a consequence of Corollaries 2.4 and 2.5, we have the following growth estimate on  $|\nabla_x u|$  when  $f = f(x)$  is Lipschitz.

**Corollary 2.6** (Boundary Growth of  $\partial_i u$ ). *Suppose  $u \in H^1(B_\lambda^+, y^a)$  satisfies (2.7) where  $f = f(x)$  and  $f \in C^{0,1}(\bar{B}_\lambda^+)$ . Then, for any  $i \in 1, \dots, n$ ,*

$$|\partial_i u(X)| \leq Cy^{2s} \quad \text{in } B_{\lambda/4}^+.$$

Here, the constant  $C > 0$  depends on  $a, n, \lambda, \|u\|_{L^\infty(B_\lambda^+)}$ , and  $\|f\|_{C^{0,1}(\bar{B}_\lambda^+)}$ .

*Proof.* For any  $i = 1, \dots, n$ , we have that  $L_a(\partial_i u) = y^a h$  where  $h := \partial_i f$ . Since  $f \in C^{0,1}(\bar{B}_\lambda^+)$ , it follows that  $h \in L^\infty(B_\lambda^+)$ . Applying Corollary 2.5 to  $\partial_i u$  implies that

$$|\partial_y(\partial_i u(X))| \leq Cy^{-a},$$

where, using Proposition 2.3, we see that  $C$  depends on  $a, n, \lambda, \|u\|_{L^\infty(B_\lambda^+)}, \|f\|_{C^{0,1}(\bar{B}_\lambda^+)}$ . Since  $\partial_i u(x, 0) = 0$ , we determine that

$$|\partial_i u(X)| \leq C \left| \int_0^y t^{2s-1} dt \right| = Cy^{2s}.$$

$\square$

We have the same growth estimate for  $|\nabla_x u|$  when  $f$  is less regular and unconstrained to depend only on  $x$  so long as it vanishes on  $\{y = 0\}$ .

**Corollary 2.7** (Boundary Growth of  $\partial_i u$ ). *Suppose  $u \in H^1(B_\lambda^+, y^a)$  satisfies (2.7) where  $f \in C(\overline{B_\lambda^+})$  and  $f$  vanishes on  $\{y = 0\}$ . If  $g := y^a \partial_y f \in L^\infty(B_\lambda^+)$ , then for any  $i \in 1, \dots, n$ ,*

$$|\partial_i u(X)| \leq C y^{2s} \quad \text{in } B_{\lambda/4}^+.$$

Here, the constant  $C > 0$  depends on  $a, n, \lambda, \|u\|_{L^\infty(B_\lambda^+)}, \|f\|_{L^\infty(B_\lambda^+)}$ , and  $\|g\|_{L^\infty(B_\lambda^+)}$ .

*Proof.* Let  $\bar{w} := |y|^a \partial_y \bar{u}$  and note that  $L_{-a} \bar{w} = |y|^{-a} \bar{g}$  in  $B_\lambda$  where  $\bar{g} := |y|^a \partial_y \bar{f} \in L^\infty(B_\lambda)$ . From Proposition 2.3, we find that

$$\|\partial_i \bar{w}\|_{L^\infty(B_{\lambda/4}^+)} \leq C \lambda^{-1} \|\bar{w}\|_{L^\infty(B_\lambda^+)} + C \|\bar{g}\|_{L^\infty(B_\lambda^+)} \quad \forall i \in 1, \dots, n.$$

In other words,

$$|\partial_y(\partial_i u(X))| \leq C y^{-a},$$

where we see from Proposition 2.3 that  $C$  depends on  $a, n, \lambda, \|u\|_{L^\infty(B_\lambda^+)}, \|f\|_{L^\infty(B_\lambda^+)}$ , and  $\|g\|_{L^\infty(B_\lambda^+)}$ . Arguing as in the proof of Corollary 2.6 completes the proof.  $\square$

### 3. A HIGHER ORDER BOUNDARY HARNACK ESTIMATE: $\Gamma \in C^{k+2, \alpha}$ FOR $k \geq 0$

In this section, we prove a pointwise higher order boundary Harnack estimate when  $\Gamma$  is at least  $C^{2, \alpha}$ . This estimate, Proposition 3.2, and its  $C^{1, \alpha}$  counterpart, Proposition 5.4, will play key roles in proving higher regularity of the free boundary in (1.1), as sketched in the introduction. We refer the reader to Section 7 for the details of how exactly this is accomplished.

Let  $U \in C(B_1)$  be even in  $y$  and normalized so that  $U(e_n/2) = 1$ . Suppose further that  $U \equiv 0$  on  $\mathcal{P}$  and  $U > 0$  on  $B_1 \setminus \mathcal{P}$ , and assume  $U$  satisfies

$$L_a U = |y|^a \left( \frac{U_a}{r} T(x, r) + G(X) \right) \quad \text{in } B_1 \setminus \mathcal{P} \quad (3.1)$$

where  $T(x, r)$  is a polynomial of degree  $k + 1$  and

$$\|T\| \leq 1 \quad \text{and} \quad |G(X)| \leq r^{s-1} |X|^{k+1+\alpha}.$$

In Proposition 4.1, we show that if  $\Gamma \in C^{k+2, \alpha}$  with  $\|\Gamma\|_{C^{k+2, \alpha}} \leq 1$  and  $U$  is as above, then  $U$  takes the form

$$U = U_a(P_0 + O(|X|^{k+1+\alpha})) \quad (3.2)$$

for some polynomial  $P_0(x, r)$  of degree  $k + 1$  with  $\|P_0\| \leq C$  and  $U_a$  as defined in (2.2). Formally, if we differentiate (3.2), we find that

$$\nabla_x U = \frac{U_a}{r} \left( s P_0 \nu + r \nabla_x P_0 + (\partial_r P_0) d\nu + O(|X|^{k+1+\alpha}) \right) \quad (3.3)$$

and

$$\nabla U \cdot \nabla r = \frac{U_a}{r} \left( s P_0 + (\partial_r P_0) r + \nabla_x P_0 \cdot (d\nu) + O(|X|^{k+1+\alpha}) \right). \quad (3.4)$$

Rigorously justifying these expansions in our application to the fractional obstacle problem will require a delicate argument, which we present in Proposition 7.1. That said, in Proposition 3.2, we simply make the assumption that (3.3) and (3.4) hold.

**Remark 3.1.** When  $T \equiv G \equiv 0$ , (3.3) and (3.4) follow by arguing as in Section 5 of [12] and the Appendix of [14], using the regularity results in Section 2.

**Proposition 3.2.** *Let  $\Gamma \in C^{k+2, \alpha}$  with  $\|\Gamma\|_{C^{k+2, \alpha}} \leq 1$ . Let  $U, T, G$ , and  $P_0$  be as in (3.1), (3.3), and (3.4). Suppose that  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(B_1)} \leq 1$ , vanishes on  $\mathcal{P}$ , and satisfies*

$$L_a u = |y|^a \left( \frac{U_a}{r} R(x, r) + F(X) \right) \quad \text{in } B_1 \setminus \mathcal{P}$$

where  $R(x, r)$  is a polynomial of degree  $k + 1$  with  $\|R\| \leq 1$  and

$$|F(X)| \leq r^{s-1} |X|^{k+1+\alpha}.$$



Then, there exists a polynomial  $P(x, r)$  of degree  $k + 2$  with  $\|P\| \leq C$  such that

$$\left| \frac{u}{U} - P \right| \leq C|X|^{k+2+\alpha}$$

for some constant  $C = C(a, n, k, \alpha) > 0$ .

Proposition 3.2 is proved via a perturbation argument that relies on following result, where we consider the specific case that  $T, G, R, F \equiv 0$  and  $\Gamma$  is flat.

**Proposition 3.3.** *Suppose  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(B_1)} \leq 1$  and satisfies*

$$\begin{cases} L_a u = 0 & \text{in } B_1 \setminus \{x_n \leq 0, y = 0\} \\ u = 0 & \text{on } \{x_n \leq 0, y = 0\}. \end{cases} \quad (3.5)$$

Then, for any  $k \geq 0$ , there exists a polynomial  $\bar{P}(x, r)$  of degree  $k$  with  $\|\bar{P}\| \leq C$  such that  $\bar{U}_a \bar{P}$  is  $a$ -harmonic in  $B_1 \setminus \{x_n \leq 0, y = 0\}$  and

$$|u - \bar{U}_a \bar{P}| \leq C \bar{U}_a |X|^{k+1} \quad (3.6)$$

for some constant  $C = C(a, n, k) > 0$ .

Recall that  $\bar{U}_a$ , defined in (2.4), is  $U_a$  when  $\Gamma$  is flat. We postpone the proof of Proposition 3.3 until Section 6. In order to proceed with proof of Proposition 3.2, we need to adapt the notion of approximating polynomials for  $u/U$ , introduced in [14], to our setting. Observe that

$$\begin{aligned} L_a(Ux^\mu r^m) &= x^\mu r^m L_a U + U L_a(x^\mu r^m) + 2|y|^a \nabla(x^\mu r^m) \cdot \nabla U \\ &= |y|^a (\text{I} + \text{II} + \text{III}) \end{aligned} \quad (3.7)$$

where, letting  $\bar{i}$  denote the  $n$ -tuple with a one in the  $i$ th position and zeros everywhere else,

$$\begin{aligned} \text{I} &= \frac{U_a}{r} x^\mu r^m T(x, r) + x^\mu r^m G(X), \\ \text{II} &= \frac{U}{r} (m(a + m - d\kappa) x^\mu r^{m-1} + \mu_i(\mu_i - 1) r^{m+1} x^{\mu-2\bar{i}} + 2dmr^{m-1} \nabla(x^\mu) \cdot \nu), \\ \text{III} &= 2(r^m \nabla_x U \cdot \nabla(x^m) + mx^\mu r^{m-1} \nabla U \cdot \nabla r). \end{aligned}$$

Up to a dilation, we can assume that

$$\|\Gamma\|_{C^{k+2, \alpha}} \leq \varepsilon, \quad \|T\|, \|R\| \leq \varepsilon, \quad \text{and} \quad |G(X)|, |F(X)| \leq \varepsilon r^{s-1} |X|^{k+1+\alpha} \quad (3.8)$$

for any  $\varepsilon > 0$ . For  $\varepsilon > 0$  sufficiently and universally small, the constant term of  $P_0$  in (3.2) is nonzero (see Remark 3.6 below). So, up to multiplication by a constant, (3.2) takes the form

$$U = U_a(1 + \varepsilon Q_0 + \varepsilon O(|X|^{k+1+\alpha})), \quad (3.9)$$

where  $Q_0(x, r)$  is a degree  $k + 1$  polynomial with zero constant term and  $\|Q_0\| \leq 1$ .

Taylor expansions of  $\nu$ ,  $\kappa$ , and  $d$  yield

$$\nu_i = \delta_{in} + \dots + \varepsilon O(|X|^{k+1+\alpha}), \quad \kappa = \kappa(0) + \dots + \varepsilon O(|X|^{k+\alpha}), \quad \text{and} \quad d = x_n + \dots + \varepsilon O(|X|^{k+2+\alpha}).$$

Hence, using (3.3) and (3.4) to expand III, we find that I, II, and III become

$$\begin{aligned} \text{I} &= \frac{U_a}{r} s_{\sigma l}^{\mu m} x^\sigma r^l + \varepsilon O(r^{s-1} |X|^{k+1+\alpha}), \\ \text{II} &= \frac{U}{r} \left( m(a + m + 2\mu_n) x^\mu r^{m-1} + \mu_i(\mu_i - 1) r^{m+1} x^{\mu-2\bar{i}} + a_{\sigma l}^{\mu m} x^\sigma r^l + \varepsilon O(|X|^{k+1+\alpha}) \right), \\ \text{III} &= \frac{U_a}{r} \left( 2sr^m \mu_n x^{\mu-\bar{n}} + 2smx^\mu r^{m-1} + b_{\sigma l}^{\mu m} x^\sigma r^l + \varepsilon O(|X|^{k+1+\alpha}) \right). \end{aligned} \quad (3.10)$$

Here,  $s_{\sigma l}^{\mu m}$ ,  $a_{\sigma l}^{\mu m}$ , and  $b_{\sigma l}^{\mu m}$  are coefficients of monomials of degree at least  $|\mu| + m$  and at most  $k + 1$ ; that is,

$$s_{\sigma l}^{\mu m}, a_{\sigma l}^{\mu m}, b_{\sigma l}^{\mu m} \neq 0 \quad \text{only if} \quad |\mu| + m \leq |\sigma| + l \leq k + 1.$$

Furthermore, since the monomials  $a_{\sigma l}^{\mu m} x^\sigma r^l$  and  $b_{\sigma l}^{\mu m} x^\sigma r^l$  come from the Taylor expansions of  $\nu, \kappa$ , and  $d$  (which vanish when  $\Gamma$  is flat), recalling (3.8), we have that

$$|s_{\sigma l}^{\mu m}|, |a_{\sigma l}^{\mu m}|, |b_{\sigma l}^{\mu m}| \leq C\varepsilon.$$

Therefore, from (3.7), (3.9), and (3.10), we determine that

$$L_a(Ux^\mu r^m) = |y|^a \left( \frac{U_a}{r} \left( mx^\mu r^{m-1} (1 + m + 2\mu_n) + 2s\mu_n r^m x^{\mu-\bar{n}} + \mu_i(\mu_i - 1)r^{m+1} x^{\mu-2\bar{i}} + c_{\sigma l}^{\mu m} x^\sigma r^l \right) + \varepsilon O(r^{s-1}|X|^{k+1+\alpha}) \right)$$

where

$$c_{\sigma l}^{\mu m} \neq 0 \quad \text{only if} \quad |\mu| + m \leq |\sigma| + l \leq k + 1 \quad \text{and} \quad |c_{\sigma l}^{\mu m}| \leq C\varepsilon.$$

Thus, given a polynomial  $P(x, r) = p_{\mu m} x^\mu r^m$  of degree  $k + 2$ ,

$$L_a(UP) = |y|^a \left( \frac{U_a}{r} A_{\sigma l} x^\sigma r^l + h(X) \right)$$

where  $|\sigma| + l \leq k + 1$ ,

$$|h(X)| \leq C\varepsilon \|P\| r^{s-1} |X|^{k+1+\alpha}, \quad (3.11)$$

and

$$A_{\sigma l} = (l + 1)(l + 2 + 2\sigma_n)p_{\sigma, l+1} + 2s(\sigma_n + 1)p_{\sigma+\bar{n}, l} + (\sigma_i + 1)(\sigma_i + 2)p_{\sigma+2\bar{i}, l-1} + c_{\sigma l}^{\mu m} p_{\mu m}. \quad (3.12)$$

From (3.12), we see that  $p_{\sigma, l+1}$  can be expressed in terms of  $A_{\sigma l}$ , a linear combination of  $p_{\mu m}$  for  $\mu + m \leq |\sigma| + l$ , and a linear combination of  $p_{\mu m}$  for  $\mu + m \leq |\sigma| + l$  and  $m \leq l$ . Consequently, the coefficients  $p_{\mu m}$  are uniquely determined by the linear system (3.12) given  $A_{\sigma l}$  and  $p_{\mu 0}$ .

**Definition 3.4.** Let  $u$  and  $U$  be as in Proposition 3.2. A polynomial  $P(x, r)$  of degree  $k + 2$  is *approximating* for  $u/U$  if the coefficients  $A_{\sigma l}$  coincide with the coefficients of  $R$ .

Before we prove Proposition 3.2, let us make two remarks and state a lemma.

**Remark 3.5.** While  $U_a$  is not  $a$ -harmonic in  $B_1 \setminus \mathcal{P}$ , it is comparable in  $B_1$  to a function  $V_a$  that is  $a$ -harmonic in  $B_1 \setminus \mathcal{P}$ . Indeed, using the upper and lower barriers  $V_\pm := (1 \pm r/2)U_a$ , one can construct such a function  $V_a$  by Perron's method.

**Remark 3.6.** Up to an initial dilation, taking  $\|\Gamma\|_{C^{k+2, \alpha}} \leq \varepsilon$  for a universally small  $\varepsilon > 0$ , the constant term of  $P_0$  in (3.2) is nonzero. If  $U$  and  $U_a$  were  $a$ -harmonic in  $B_1 \setminus \mathcal{P}$ , this would follow directly from the boundary Harnack estimate applied to  $U_a/U$  without a dilation. By Remark 3.5,  $U_a$  is comparable to the  $a$ -harmonic function  $V_a$ . For  $\varepsilon > 0$  sufficiently small (universally so), we will find that  $U$  is also comparable to an  $a$ -harmonic function  $W$ ; hence, we can effectively apply the boundary Harnack estimate to  $U_a/U$  passing through the quotient  $V_a/W$  to conclude. More specifically, after dilating, let us normalize so that  $U(e_n/2) = 1$ . First, let  $W$  satisfy

$$\begin{cases} L_a W = 0 & \text{in } B_1 \setminus \mathcal{P} \\ W = U & \text{on } \partial B_1 \cup \mathcal{P}. \end{cases}$$

The strong maximum principle ensures that  $W$  is positive in  $B_1 \setminus \mathcal{P}$ . Applying the boundary Harnack estimate to  $W$  and  $V_a$ , Remark 3.5 implies that

$$cU_a \leq \frac{W}{W(e_n/2)} \leq CU_a.$$

Second, let  $V := U - W$ , and observe that

$$\begin{cases} |L_a V| \leq C\varepsilon |y|^a r^{s-1} & \text{in } B_1 \setminus \mathcal{P} \\ V = 0 & \text{on } \partial B_1 \cup \mathcal{P}. \end{cases}$$

Lemma 3.7 then shows that

$$|V| \leq C\varepsilon U_a.$$

So,  $1 - C\varepsilon \leq W(e_n/2) \leq 1 + C\varepsilon$  and for  $\varepsilon > 0$  small,

$$cU_a \leq U \leq CU_a.$$

Now, if the constant term of  $P_0$  were zero, then (3.2) would yield  $cU_a \leq U \leq CU_a|X|$ , which is impossible.

In addition to its use in Remark 3.6 above, the following lemma will be used at several other points. The proof follows by considering the upper and lower barriers  $v_{\pm} := \pm C(U_a - U_a^{1/s})$ .

**Lemma 3.7.** *Let  $v \in C(B_1)$  satisfy*

$$\begin{cases} |L_a v| \leq |y|^{a_r s-1} & \text{in } B_1 \setminus \mathcal{P} \\ v = 0 & \text{on } \partial B_1 \cup \mathcal{P}. \end{cases}$$

Then,

$$|v| \leq CU_a$$

for some constant  $C = C(a) > 0$ .

We are now in a position to prove Proposition 3.2.

*Proof of Proposition 3.2.* First, let  $\varepsilon > 0$  in (3.8) be sufficiently small so that Remark 3.6 holds. Then, solving a system of linear equations as discussed above, we obtain an initial approximating polynomial  $Q^0(x, r)$  of degree  $k+2$  for  $u/U$ . Up to multiplying  $u$  by a small constant and further decreasing  $\varepsilon > 0$  (recall that  $\|Q^0\| \leq C\varepsilon$ ), we can assume that

$$\|Q^0\| \leq 1, \quad \|u - UQ^0\|_{L^\infty(B_1)} \leq 1, \quad \text{and} \quad |L_a[u - UQ^0](X)| \leq |y|^a \varepsilon r^{s-1} |X|^{k+1+\alpha}.$$

*Step 1:* There exists  $0 < \rho < 1$ , depending on  $a, n, k$ , and  $\alpha$ , such that, up to further decreasing  $\varepsilon > 0$ , the following holds. If there exists a polynomial  $Q$  of degree  $k+2$  that is approximating for  $u/U$  with  $\|Q\| \leq 1$  and

$$\|u - UQ\|_{L^\infty(B_\lambda)} \leq \lambda^{k+2+\alpha+s},$$

then there exists a polynomial  $Q'$  of degree  $k+2$  that is approximating for  $u/U$  with

$$\|u - UQ'\|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{k+2+\alpha+s}$$

and

$$\|Q' - Q\|_{L^\infty(B_\lambda)} \leq C\lambda^{k+2+\alpha}.$$

for some constant  $C = C(a, n, k, \alpha) > 0$ .

For any  $\lambda > 0$ , define the rescalings

$$\mathcal{P}_\lambda := \frac{1}{\lambda}\mathcal{P}, \quad r_\lambda(X) := \frac{r(\lambda X)}{\lambda}, \quad U_{a,\lambda}(X) := \frac{U_a(\lambda X)}{\lambda^s}, \quad U_\lambda(X) := \frac{U(\lambda X)}{\lambda^s}, \quad (3.13)$$

and

$$\tilde{u}(X) := \frac{[u - UQ](\lambda X)}{\lambda^{k+2+\alpha+s}}.$$

Thus,  $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$ , and by (3.11), we have that

$$|L_a \tilde{u}(X)| \leq C\varepsilon |y|^a r_\lambda^{s-1} |X|^{k+1+\alpha}. \quad (3.14)$$

Let  $w$  be the unique solution to

$$\begin{cases} L_a w = 0 & \text{in } B_1 \setminus \mathcal{P}_\lambda \\ w = 0 & \text{on } \mathcal{P}_\lambda \\ w = \tilde{u} & \text{on } \partial B_1. \end{cases} \quad (3.15)$$

Notice that  $w$  is even in  $y$  by the symmetry of the domain and boundary data and  $\|w\|_{L^\infty(B_1)} \leq 1$  by the maximum principle. Since  $\mathcal{P}_\lambda$  has uniformly positive  $L_a$ -capacity independently of  $\varepsilon$  and  $\lambda$ ,  $w$

is uniformly Hölder continuous in compact subsets of  $B_1$ . So, letting  $\bar{w}$  be the solution of (3.15) in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ , by compactness,  $w$  is uniformly close to  $\bar{w}$  if  $\varepsilon$  is sufficiently small (universally so). Indeed, recall that  $\Gamma \rightarrow \{x_n = 0, y = 0\}$  in  $C^{k+2,\alpha}$  as  $\varepsilon \rightarrow 0$ . Furthermore, thanks to (3.9), we have that  $U_\lambda \rightarrow \bar{U}_a$  uniformly as  $\varepsilon \rightarrow 0$ .

Proposition 3.3 ensures that there exists a polynomial

$$\bar{P}(X) := \bar{p}_{\mu m} x^\mu |z|^m$$

of degree  $k+2$  such that  $\|\bar{P}\| \leq C$ ,  $\bar{U}_a \bar{P}$  is  $a$ -harmonic in the set  $B_1 \setminus \{x_n \leq 0, y = 0\}$ , and

$$\|\bar{w} - \bar{U}_a \bar{P}\|_{L^\infty(B_\rho)} \leq C \rho^{k+3+s}.$$

Notice that the  $a$ -harmonicity of  $\bar{U}_a \bar{P}$  implies that

$$(l+1)(l+2+2\sigma_n)\bar{p}_{\sigma,l+1} + 2s(\sigma_n+1)\bar{p}_{\sigma+\bar{n},l} + (\sigma_i+1)(\sigma_i+2)\bar{p}_{\sigma+2\bar{i},l-1} = 0 \quad \forall(\sigma,l). \quad (3.16)$$

Therefore, choosing  $\rho$  and then  $\varepsilon$  sufficiently small depending on  $a, n, k$ , and  $\alpha$ , we find that

$$\begin{aligned} \|w - U_\lambda \tilde{P}\|_{L^\infty(B_\rho)} &\leq \|w - \bar{w}\|_{L^\infty(B_{1/2})} + \|U_\lambda \tilde{P} - \bar{U}_a \bar{P}\|_{L^\infty(B_{1/2})} + \|\bar{w} - \bar{U}_a \bar{P}\|_{L^\infty(B_\rho)} \\ &\leq \frac{1}{4} \rho^{k+2+\alpha+s} \end{aligned} \quad (3.17)$$

where  $\tilde{P}(X) := \bar{p}_{\mu m} x^\mu r_\lambda^m$  has the same coefficients as  $\bar{P}$ . Now, set  $v := \tilde{u} - w$ . From (3.14), we find that

$$\begin{cases} |L_a v| \leq \varepsilon |y|^a r_\lambda^{s-1} & \text{in } B_1 \setminus \mathcal{P}_\lambda \\ v = 0 & \text{on } \partial B_1 \cup \mathcal{P}_\lambda. \end{cases}$$

From Lemma 3.7 and (3.9), we deduce that

$$|v| \leq C\varepsilon U_{a,\lambda} \leq C\varepsilon U_\lambda. \quad (3.18)$$

Then, combining (3.18) and (3.17) and further decreasing  $\varepsilon$  depending on  $\rho, a, n, k$ , and  $\alpha$ , we find that

$$\|\tilde{u} - U_\lambda \tilde{P}\|_{L^\infty(B_\rho)} \leq \|v\|_{L^\infty(B_\rho)} + \|w - U_\lambda \tilde{P}\|_{L^\infty(B_\rho)} \leq \frac{1}{2} \rho^{k+2+\alpha+s}.$$

Rescaling implies that

$$\|u - U\tilde{Q}\|_{L^\infty(B_{\rho\lambda})} \leq \frac{1}{2} (\rho\lambda)^{k+2+\alpha+s}$$

with  $\tilde{Q}(X) := Q(X) + \lambda^{k+2+\alpha} \tilde{P}(X/\lambda)$ .

To conclude, we must alter  $\tilde{Q}$  to make it an approximating polynomial for  $u/U$  by replacing  $\tilde{P}(X/\lambda)$  with another polynomial  $P'(X/\lambda)$ . As  $Q$  is already an approximating polynomial for  $u/U$ , we need the coefficients  $p'_{\mu m}$  of  $P'$  to satisfy the system

$$(l+1)(l+2+2\sigma_n)p'_{\sigma,l+1} + 2s(\sigma_n+1)p'_{\sigma+\bar{n},l} + (\sigma_i+1)(\sigma_i+2)p'_{\sigma+2\bar{i},l-1} + \tilde{c}_{\sigma l}^{\mu m} p'_{\mu m} = 0 \quad \forall(\sigma,l) \quad (3.19)$$

where

$$\tilde{c}_{\sigma l}^{\mu m} := \lambda^{|\sigma|+l+1-|\mu|-m} c_{\sigma l}^{\mu m}.$$

Notice that  $|\tilde{c}_{\sigma l}^{\mu m}| \leq |c_{\sigma l}^{\mu m}| \leq C\varepsilon$ . Furthermore, subtracting (3.19) from (3.16), we see that  $P' - \tilde{P}$  solves the system (3.19) with right-hand side

$$A_{\sigma l} = \tilde{c}_{\sigma l}^{\mu m} \bar{p}_{\mu m}.$$

Hence,  $|A_{\sigma l}| \leq C\varepsilon$ , and choosing  $p'_{\mu 0} = \bar{p}_{\mu 0}$ , we uniquely determine  $P'$  and find that

$$\|P' - \tilde{P}\| \leq C\varepsilon.$$

Setting  $Q'(X) := Q(X) + \lambda^{k+2+\alpha} P'(X/\lambda)$  concludes Step 1.

*Step 2: Iteration and Upgrade.*

Iterating Step 1, letting  $\lambda = \rho^j$  for  $j = 0, 1, 2, \dots$ , we find a limiting approximating polynomial  $P$  such that  $\|P\| \leq C$  and

$$\|u - UP\|_{L^\infty(B_{\rho^j})} \leq C\rho^{j(k+2+\alpha+s)} \quad \forall j \in \mathbb{N}.$$

To upgrade this inequality to

$$|u - UP| \leq CU|X|^{k+2+\alpha}, \quad (3.20)$$

we argue as in Step 1 in  $B_1 \setminus \mathcal{P}_\lambda$ . Setting

$$\tilde{u}(X) := \frac{[u - UP](\lambda X)}{\lambda^{k+2+\alpha+s}},$$

we have that

$$|\tilde{u}| \leq |w| + |v| \leq CU_{a,\lambda} \leq CU_\lambda \quad \text{in } B_{1/2}.$$

Indeed, that  $v$  and  $w$  are controlled by  $U_{a,\lambda}$  comes from Lemma 3.7 and an application of the boundary Harnack estimate (cf. Remark 3.5), while the last inequality comes from (3.9). Thus, after rescaling, we deduce that (3.20) holds since  $0 < \lambda \leq 1$  was arbitrary.  $\square$

Keeping Remark 3.1 in mind, if  $U$  is  $a$ -harmonic, then (3.3) and (3.4) hold. So, a consequence of Proposition 3.2 and Proposition 5.4, its  $C^{1,\alpha}$  analogue, is the following full generalization of [14, Theorem 2.3].

**Theorem 3.8.** *Suppose  $0 \in \Gamma := \partial_{\mathbb{R}^n} \mathcal{P}$  with  $\Gamma \in C^{k,\alpha}$  for  $k \geq 1$  and  $\|\Gamma\|_{C^{k,\alpha}} \leq 1$ . If  $u$  and  $U$  are even in  $x_{n+1}$ ,  $\|u\|_{L^\infty(B_1)} \leq 1$ ,*

$$\begin{cases} L_a u = |y|^a \frac{U_a}{r} f & \text{in } B_1 \setminus \mathcal{P} \\ u = 0 & \text{on } \mathcal{P} \end{cases}$$

for

$$f \in C_{x,r}^{k-1,\alpha}(\Gamma \cap B_1) \quad \text{with} \quad \|f\|_{C_{x,r}^{k-1,\alpha}(\Gamma \cap B_1)} \leq 1,$$

and  $U > 0$  is  $a$ -harmonic in  $B_1 \setminus \mathcal{P}$  with  $U(\nu(0)/2) = 1$ , where  $\nu$  is the outer unit normal to  $\mathcal{P}$ , then

$$\left\| \frac{u}{U} \right\|_{C_{x,r}^{k,\alpha}(\Gamma \cap B_{1/2})} \leq C$$

for some  $C = C(a, n, k, \alpha) > 0$ .

#### 4. SCHAUDER ESTIMATES: $\Gamma \in C^{k+2,\alpha}$ FOR $k \geq 0$

In Proposition 3.2, we were crucially able to approximate  $U$  in terms of  $U_a P_0$ , where  $P_0 = P_0(x, r)$  is a polynomial of degree  $k + 1$ . This approximation, (3.2), is a consequence of the Schauder estimates of Proposition 4.1 below. These Schauder estimates roughly say if  $u$  satisfies

$$L_a u = |y|^a \frac{U_a}{r} f \quad \text{in } B_1 \setminus \mathcal{P} \quad \text{and} \quad u = 0 \quad \text{on } \mathcal{P},$$

then  $u$  gains regularity in terms of the regularity of  $f$  and  $\Gamma$ . More precisely, we find that  $u/U_a \in C_{x,r}^{k+1,\alpha}(0)$  if  $f \in C_{x,r}^{k,\alpha}(0)$  and  $\Gamma \in C^{k+2,\alpha}$ .

**Proposition 4.1.** *Let  $\Gamma \in C^{k+2,\alpha}$  with  $\|\Gamma\|_{C^{k+2,\alpha}} \leq 1$ . Suppose  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(B_1)} \leq 1$ , vanishes on  $\mathcal{P}$ , and satisfies*

$$L_a u(X) = |y|^a \left( \frac{U_a}{r} R(x, r) + F(X) \right) \quad \text{in } B_1 \setminus \mathcal{P}$$

where  $R(x, r)$  is a polynomial of degree  $k$  with  $\|R\| \leq 1$  and

$$|F(X)| \leq r^{s-1} |X|^{k+\alpha}.$$

Then, there exists a polynomial  $P_0(x, r)$  of degree  $k + 1$  with  $\|P_0\| \leq C$  such that

$$|u - U_a P_0| \leq CU_a |X|^{k+1+\alpha}$$

and

$$|L_a(u - U_a P_0)| \leq C|y|^a r^{s-1} |X|^{k+\alpha} \quad \text{in } B_1 \setminus \mathcal{P}$$

for some constant  $C = C(a, n, k, \alpha) > 0$ .

To prove Proposition 4.1, we must first extend the appropriate notion of approximating polynomial to this setting. We compute that

$$\begin{aligned} L_a(U_a x^\mu r^m) &= |y|^a \frac{U_a}{r} \left( - (dm + sr)\kappa x^\mu r^{m-1} + m(m+1)x^\mu r^{m-1} \right. \\ &\quad \left. + 2r^{m-1}(dm + sr)\nu \cdot \nabla_x x^\mu + \mu_i(\mu_i - 1)x^{\mu-2\bar{i}} r^{m+1} \right). \end{aligned}$$

Each of the functions  $\nu_i, \kappa$ , and  $d$  can be written as the sum of a degree  $k$  polynomial in  $x$  and a  $C^{k,\alpha}$  function in  $x$  whose derivatives vanish up to order  $k$ . The lowest degree terms in the Taylor expansion at zero of  $\nu_i, \kappa$ , and  $d$  are  $\delta_{in}, \kappa(0)$ , and  $x_n$  respectively. Hence, grouping terms by degree up to order  $k$  and the remainder, we see that

$$\begin{aligned} L_a(U_a x^\mu r^m) &= |y|^a \frac{U_a}{r} \left( m(m+1 + 2\mu_n)x^\mu r^{m-1} + 2s\mu_n x^{\mu-\bar{n}} r^m \right. \\ &\quad \left. + \mu_i(\mu_i - 1)x^{\mu-2\bar{i}} r^{m+1} + c_{\sigma l}^{\mu m} x^\sigma r^l + h^{\mu m}(x, r) \right). \end{aligned}$$

Here,  $c_{\sigma l}^{\mu m} \neq 0$  only if  $|\mu| + m \leq |\sigma| + l \leq k$ . Also,

$$h^{\mu m}(x, r) := r^m h_m^\mu(x) + m r^{m-1} h_{m-1}^\mu(x),$$

and  $h_m^\mu, h_{m-1}^\mu \in C^{k,\alpha}(B_1^*)$  have vanishing derivatives up to order  $k - m$  and  $k - (m - 1)$  at zero respectively. The coefficients  $c_{\sigma l}^{\mu m}$  are all linear combinations of the Taylor coefficients at the origin of  $\kappa, d\kappa, \nu_i$ , and  $d\nu_i$ , which vanish if  $\Gamma$  is flat. After a dilation making  $\|\Gamma\|_{C^{k+2,\alpha}} \leq \varepsilon$ , we may assume that

$$|c_{\sigma l}^{\mu m}| \leq \varepsilon, \quad \|h_m^\mu\|_{C^{k,\alpha}(B_1^*)} \leq \varepsilon, \quad \text{and} \quad \|h_{m-1}^\mu\|_{C^{k,\alpha}(B_1^*)} \leq \varepsilon.$$

Therefore, if  $P(x, r) = p_{\mu m} x^\mu r^m$  is a polynomial of degree  $k + 1$ , then

$$L_a(U_a P) = |y|^a \frac{U_a}{r} \left( A_{\sigma l} x^\sigma r^l + h(x, r) \right)$$

where  $|\sigma| + l \leq k$ ,

$$A_{\sigma l} = (l+1)(l+2+2\sigma_n)p_{\sigma, l+1} + 2s(\sigma_n+1)p_{\sigma+\bar{n}, l} + (\sigma_i+1)(\sigma_i+2)p_{\sigma+2\bar{i}, l-1} + c_{\sigma l}^{\mu m} p_{\mu m}, \quad (4.1)$$

and

$$h(x, r) := \sum_{m=0}^k r^m h_m(x)$$

for  $h_m \in C^{k,\alpha}(B_1^*)$  with vanishing derivatives up to order  $k - m$  at zero. Assuming that  $\|\Gamma\|_{C^{k+2,\alpha}} \leq \varepsilon$ , we have

$$|h(X)| \leq \varepsilon \|P\| |X|^{k+\alpha}.$$

Considering (4.1), we see that  $p_{\sigma, l+1}$  can be expressed in terms of  $A_{\sigma l}$ , a linear combination of  $p_{\mu m}$  for  $\mu + m \leq |\sigma| + l$ , and a linear combination of  $p_{\mu m}$  for  $\mu + m \leq |\sigma| + l$  and  $m \leq l$ . Thus, the coefficients  $p_{\mu m}$  are uniquely determined by the linear system (4.1) given  $A_{\sigma l}$  and  $p_{\mu 0}$ .

**Definition 4.2.** Let  $u$  be as in Proposition 4.1. We call a polynomial  $P(x, r)$  of degree  $k + 1$  *approximating* for  $u/U_a$  if the coefficients  $A_{\sigma l}$  coincide with the coefficients of  $R$ .

**Remark 4.3.** Observe that

$$L_a U_a = -|y|^a \frac{U_a}{r} s\kappa(x) \quad \text{and} \quad L_a U = |y|^a \left( \frac{U_a}{r} T(x, r) + G(X) \right).$$

Since  $\Gamma \in C^{k+2,\alpha}$ , the mean curvature  $\kappa = \kappa(x)$  does not possess enough regularity to yield the same order error as  $G$  after being expanded. Indeed, letting  $g := \kappa - T$  where  $T$  is the  $k$ th order Taylor polynomial of  $\kappa$  at the origin, we see that

$$\frac{U_a}{r}|g(X)| \leq r^{s-1}|X|^{k+\alpha} \quad \text{while} \quad |G(X)| \leq r^{s-1}|X|^{k+1+\alpha}.$$

This discrepancy lies at the heart of the difference in approximating  $u/U_a$  and  $u/U$ .

With the correct notion of approximating polynomial in hand, the proof of Proposition 4.1 is now identical to that of Proposition 3.2 upon replacing  $U$  with  $U_a$ ; it is therefore omitted.

### 5. THE LOW REGULARITY CASE: $\Gamma \in C^{1,\alpha}$

The goal of this section is to prove Proposition 5.4, which extends the higher order boundary Harnack estimate of Proposition 3.2 to the case when  $\Gamma$  is only of class  $C^{1,\alpha}$ . In this case, the functions  $r$  and  $U_a$  introduced in Section 2.1 do not possess enough regularity to directly extend the proof of Proposition 4.1 or the notion of approximating polynomial for  $u/U$  in Definition 3.4. Following [14], this is rectified by working with regularizations of  $r$  and  $U_a$ , denoted by  $r_*$  and  $U_{a,*}$  respectively. The following lemma contains estimates which will allow us to replace  $r$  and  $U_a$  by their regularizations when needed. The construction and the proofs of these estimates can be found in the Appendix.

**Lemma 5.1.** *Let  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ . There exist smooth functions  $r_*$  and  $U_{a,*}$  such that*

$$\begin{aligned} \left| \frac{r_*}{r} - 1 \right| &\leq C_* r^\alpha, & \left| \frac{U_{a,*}}{U_a} - 1 \right| &\leq C_* r^\alpha, \\ |\nabla r_* - \nabla r| &\leq C_* r^\alpha, & |\partial_y r_* - \partial_y r| &\leq C_* |y|^a U_a r^{s-1+\alpha}, & \left| \frac{|\nabla U_{a,*}|}{|\nabla U_a|} - 1 \right| &\leq C_* r^\alpha, \\ \left| L_a r_* - \frac{2(1-s)|y|^a}{r} \right| &\leq C_* |y|^a r^{\alpha-1}, & \text{and} & & |L_a U_{a,*}| &\leq C_* |y|^a r^{s-2+\alpha} \end{aligned}$$

where  $C_* = C_*(a, n, \alpha) > 0$ . If  $\|\Gamma\|_{C^{1,\alpha}} \leq \varepsilon$ , then each inequality holds with the right-hand side multiplied by  $\varepsilon$ .

The following pointwise Schauder estimate plays the role of Proposition 4.1 in the case when  $\Gamma$  is  $C^{1,\alpha}$ .

**Proposition 5.2.** *Let  $\Gamma \in C^{1,\alpha}$  with  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ . Suppose  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(B_1)} \leq 1$ , vanishes on  $\mathcal{P}$ , and satisfies*

$$|L_a u| \leq |y|^a r^{s-2+\alpha} \quad \text{in } B_1 \setminus \mathcal{P}. \quad (5.1)$$

Then, there exists a constant  $p'$  with  $|p'| \leq C$  such that

$$|u - p'U_a| \leq CU_a |X|^\alpha$$

for some constant  $C = C(a, n, \alpha) > 0$ .

Note that even though Proposition 5.2 is stated just at the origin, it holds uniformly at all points  $\Gamma \cap B_{1/2}$  since the assumption on the right-hand side in (5.1) does not distinguish the origin. The proof of Proposition 5.2 is quite similar to that of Propositions 3.2 and 4.1, but we include it to demonstrate how  $U_{a,*}$  is used. In the proof, we will make use of the following lemma, whose proof via a barrier argument is given in the Appendix.

**Lemma 5.3.** *Assume  $\|\Gamma\|_{C^{1,\alpha}} \leq \varepsilon$  with  $\alpha \in (0, 1-s)$ , and suppose  $u$  satisfies*

$$\begin{cases} |L_a u| \leq |y|^a r^{\alpha-2+s} & \text{in } B_1 \setminus \mathcal{P} \\ u = 0 & \text{on } \partial B_1 \cup \mathcal{P}. \end{cases}$$

If  $\varepsilon > 0$  is sufficiently small, depending on  $a$ ,  $n$ , and  $\alpha$ , then

$$|u| \leq CU_a$$

for some  $C = C(a, n, \alpha) > 0$ .

*Proof of Proposition 5.2.* Up to a dilation, we may assume that

$$\|\Gamma\|_{C^{1,\alpha}} \leq \varepsilon \quad \text{and} \quad |L_a u| \leq \varepsilon |y|^{a r^{s-2+\alpha}}$$

for any  $\varepsilon > 0$ , in particular, for  $\varepsilon$  small enough to apply Lemma 5.3.

*Step 1:* There exists  $0 < \rho < 1$ , depending on  $a, n$ , and  $\alpha$ , such that, up to further decreasing  $\varepsilon > 0$ , the following holds. If there exists a constant  $q$  such that  $|q| \leq 1$  and

$$\|u - qU_a\|_{L^\infty(B_\lambda)} \leq \lambda^{\alpha+s},$$

then there exists a constant  $q'$  with  $|q'| \leq C$  such that

$$\|u - Uq'\|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{\alpha+s}$$

and

$$|q' - q| \leq C\lambda^\alpha$$

for some constant  $C = C(a, n, \alpha) > 0$ .

Define  $\mathcal{P}_\lambda, r_\lambda$ , and  $U_{a,\lambda}$  as in (3.13), and consider the rescaling

$$\tilde{u}(X) := \frac{[u - qU_{a,*}](\lambda X)}{2C_*\lambda^{\alpha+s}}.$$

Note that  $\|\tilde{u}\|_{L^\infty(B_1)} \leq 1$  by Lemma 5.1. Let  $w$  be the unique solution of

$$\begin{cases} L_a w = 0 & \text{in } B_1 \setminus \mathcal{P} \\ w = 0 & \text{on } \mathcal{P} \\ w = \tilde{u} & \text{on } \partial B_1. \end{cases}$$

Observe that  $w$  is even in  $y$  and  $\|w\|_{L^\infty(B_1)} \leq 1$ . By compactness,  $w \rightarrow \bar{w}$  locally uniformly as  $\varepsilon \rightarrow 0$  where  $\bar{w}$  vanishes on  $\{x_n \leq 0, y = 0\}$  and is such that  $L_a \bar{w} = 0$  in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ . Proposition 3.3 ensures the existence of a constant  $\bar{p}$  with  $|\bar{p}| \leq C$  such that, choosing  $\rho$  and then  $\varepsilon$  sufficiently small, depending on  $a, n$ , and  $\alpha$ ,

$$\|w - \bar{p}U_{a,\lambda}\|_{L^\infty(B_\rho)} \leq \|w - \bar{p}\bar{U}_a\|_{L^\infty(B_\rho)} + \|\bar{p}(\bar{U}_a - U_{a,\lambda})\|_{L^\infty(B_\rho)} \leq \frac{1}{8C_*}\rho^{\alpha+s}.$$

Since  $v := \tilde{u} - w$  satisfies

$$\begin{cases} |L_a v| \leq \varepsilon |y|^{a r_\lambda^{s-2+\alpha}} & \text{in } B_1 \\ v = 0 & \text{on } \partial B_1 \cup \mathcal{P}_\lambda, \end{cases}$$

Lemma 5.3 shows that  $|v| \leq C\varepsilon U_{a,\lambda}$ . Then, up to further decreasing  $\varepsilon$ , we deduce that

$$\|\tilde{u} - \bar{p}U_{a,\lambda}\|_{L^\infty(B_\rho)} \leq \frac{1}{4C_*}\rho^{\alpha+s}.$$

In terms of  $u$ , this implies that

$$\|u - qU_{a,*} - 2C_*\lambda^{\alpha+s}\bar{p}U_a\|_{L^\infty(B_{\rho\lambda})} \leq \frac{1}{2}(\rho\lambda)^{\alpha+s}.$$

Consequently, by Lemma 5.1, further decreasing  $\varepsilon$  if necessary, we find that

$$\|u - q'U_a\|_{L^\infty(B_{\rho\lambda})} \leq (\rho\lambda)^{\alpha+s} \quad \text{and} \quad |q' - q| \leq C\lambda^{\alpha+s}$$

where  $q' := q + 2C_*\lambda^{\alpha+s}\bar{p}$ .

*Step 2: Iteration and Upgrade.*

Iterating Step 1, letting  $\lambda = \rho^j$ , we find that there exists a limiting constant  $p'$  such that

$$\|u - p'U_a\|_{L^\infty(B_{\rho^j})} \leq C\rho^{j(\alpha+s)} \quad \forall j \in \mathbb{N}.$$



To conclude, we must upgrade this inequality to

$$|u - p'U_a| \leq CU_a|X|^\alpha.$$

Arguing as in Step 1, in  $B_1 \setminus \mathcal{P}_\lambda$ , with

$$\tilde{u}(X) := \frac{[u - p'U_{a,*}](\lambda X)}{\lambda^{\alpha+s}},$$

we have that

$$|\tilde{u}| \leq |w| + |v| \leq CU_{a,\lambda} \quad \text{in } B_{1/2}.$$

Indeed, the bound on  $v$  comes from Lemma 5.3, and the bound on  $w$  comes from an application of the boundary Harnack estimate (see Remark 3.5) and Lemma 5.1.<sup>5</sup> Thus, after rescaling, since  $0 < \lambda \leq 1$  was arbitrary, we find that

$$|u - p'U_a| \leq |p'U_a - p'U_{a,*}| + |u - p'U_{a,*}| \leq CU_a|X|^\alpha,$$

as desired.  $\square$

We now proceed with the higher order boundary Harnack estimate. Let  $U \in C(B_1)$  be even in  $y$  with  $U \equiv 0$  on  $\mathcal{P}$  and  $U > 0$  on  $B_1 \setminus \mathcal{P}$ , normalized so that  $U(e_n/2) = 1$ , and satisfy

$$L_a U = |y|^\alpha \left( t \frac{U_a}{r} + G(X) \right) \quad \text{in } B_1 \setminus \mathcal{P} \quad (5.2)$$

where  $t$  is a constant with

$$|t| \leq 1 \quad \text{and} \quad |G(X)| \leq r^{s-1}|X|^\alpha. \quad (5.3)$$

If  $\Gamma \in C^{1,\alpha}$  with  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ , then Proposition 5.2 implies that

$$U = U_a(p' + O(|X|^\alpha)) \quad (5.4)$$

for a constant  $|p'| \leq C$ . As before, formally differentiating (5.4) yields

$$|\nabla_x U - p' \nabla_x U_a| \leq C \frac{U_a}{r} |X|^\alpha \quad (5.5)$$

and

$$|\partial_y U - p' \partial_y U_a| \leq C |y|^{-a} r^{-s} |X|^\alpha; \quad (5.6)$$

the justification of these derivative estimates for our application is somewhat delicate and is given in Proposition 7.3. Again, in the simplest case, taking  $t \equiv G \equiv 0$ , these derivative estimates can be shown by arguing as in Section 5 of [12] and the Appendix of [14], using the regularity results in Section 2 (cf. Remark 3.1).

**Proposition 5.4.** *Let  $\Gamma \in C^{1,\alpha}$  with  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ . Let  $U, t, G$ , and  $p'$  be as in (5.2), (5.3), (5.5), and (5.6). Suppose that  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(B_1)} \leq 1$ , vanishes on  $\mathcal{P}$ , and satisfies*

$$L_a u = |y|^\alpha \left( b \frac{U_a}{r} + F(X) \right) \quad \text{in } B_1 \setminus \mathcal{P}$$

where  $b$  is a constant such that  $|b| \leq 1$  and

$$|F(X)| \leq r^{s-1}|X|^\alpha.$$

Then, there exists a polynomial  $P(x, r)$  of degree 1 with  $\|P\| \leq C$  such that

$$\left| \frac{u}{U} - P \right| \leq C |X|^{1+\alpha}$$

for some constant  $C = C(a, n, \alpha) > 0$ .

<sup>5</sup> Here, we use the upper and lower barriers  $U_{a,*} \mp U_{a,*}^{1+\alpha/s}$  in Perron's method to build an  $a$ -harmonic function that vanishes on  $\mathcal{P}$  and is comparable to  $U_{a,*}$  and  $U_a$ .

To prove Proposition 5.4, we extend the notion of approximating polynomial to this low regularity setting by considering polynomials in  $(x, r_*)$  rather than in  $(x, r)$ ; that is,  $P(x, r_*) = p_0 + p_i x_i + p_{n+1} r_*$ . After performing an initial dilation, as before, using Lemma 5.1, (5.5), and (5.6), one can show that

$$L_a(UP) = |y|^a \frac{U_a}{r} (tp_0 + 2sp_n + 2p_{n+1}) + h(X)$$

with

$$|h(X)| \leq \varepsilon \|P\| r^{s-1} |X|^\alpha.$$

**Definition 5.5.** Let  $u$  and  $U$  be as in Proposition 5.4. A polynomial  $P(x, r_*)$  of degree 1 is *approximating* for  $u/U$  if

$$b = tp_0 + 2sp_n + 2p_{n+1}.$$

With this definition of approximating polynomial, the proof of Proposition 5.4 is identical to that of Proposition 3.2 and is therefore omitted.

## 6. PROOF OF PROPOSITION 3.3

In this section, we prove Proposition 3.3. That is, if  $u$  is  $a$ -harmonic in  $B_1 \setminus \{x_n \leq 0, y = 0\}$  and vanishes continuously on  $\{x_n \leq 0, y = 0\}$ , then the quotient  $u/U_a$  is  $C_{x,r}^\infty(\Gamma \cap B_{1/2})$ . The perturbative arguments of Sections 3 through 5 all rely on this core regularity result. The idea of the proof is the following. The domain  $B_1 \setminus \{x_n \leq 0, y = 0\}$  and the operator  $L_a$  are translation invariant in the  $e_i$  direction for any  $i = 1, \dots, n-1$ , so differentiating the equation  $L_a u = 0$  shows that  $u$  is smooth in these directions. We can then reduce the proof of Proposition 3.3 to the two-dimensional case, but with a right-hand side. A final reduction (Lemma 6.7) leaves us with the main task of this section, which is proving Proposition 3.3 in the case  $n = 2$  with zero right-hand side. This is Proposition 6.1 below.

It will be convenient to fix the following additional notation. For  $x' \in \mathbb{R}^{n-1}$ , we let

$$D_{\lambda, x'} := \{(x', z) \in \mathbb{R}^{n+1} : |z| < \lambda\}.$$

We sometimes suppress the dependence on  $x'$  and view  $D_{\lambda, x'} = D_\lambda$  as a subset of  $\mathbb{R}^2$ .

**Proposition 6.1.** *Let  $u \in C(B_1)$  be even in  $y$  with  $\|u\|_{L^\infty(D_1)} \leq 1$  and satisfy*

$$\begin{cases} L_a u = 0 & \text{in } D_1 \setminus \{x \leq 0, y = 0\} \\ u = 0 & \text{on } \{x \leq 0, y = 0\}. \end{cases}$$

*Then, for any  $k \geq 0$ , there exists a polynomial  $P(x, r)$  of degree  $k$  with  $\|P\| \leq C$  such that  $U_a P$  is  $a$ -harmonic in  $D_1 \setminus \{x \leq 0, y = 0\}$  and*

$$|u - U_a P| \leq C U_a |z|^{k+1}$$

*for some constant  $C = C(a, k) > 0$ .*

The geometry of our domain  $D_1 \setminus \{x \leq 0, y = 0\}$  is simplified through the change of coordinates

$$x(z_1, z_2) = z_1^2 - z_2^2 \quad \text{and} \quad y(z_1, z_2) = 2z_1 z_2,$$

which identifies the right-half unit disk  $D_1^+ := \{z \in \mathbb{R}^2 : |z| < 1, z_1 > 0\}$  and  $D_1 \setminus \{x \leq 0, y = 0\}$ .<sup>6</sup> If we let  $\bar{u}$  denote  $u$  after this change of coordinates, then  $\bar{u}$  is even in  $z_2$  and

$$\begin{cases} \bar{L}_a \bar{u} = 0 & \text{in } D_1^+ \\ \bar{u} = 0 & \text{on } \{z_1 = 0\} \end{cases}$$

where the operator  $\bar{L}_a$  (which is  $L_a$  in these coordinates) is given by

$$\bar{L}_a u := \frac{1}{4|z|^2} \operatorname{div}(|2z_1 z_2|^a \nabla u).$$

<sup>6</sup>This can be seen as the complex change of coordinates  $z \mapsto z^2$ , i.e.,  $\bar{u}(z) = u(z^2)$ . Abusing notation, we let  $z$  denote points in this new coordinate system:  $z = (z_1, z_2)$ . Similarly, we set  $D_\lambda := \{z \in \mathbb{R}^2 : |z| < \lambda\}$ .

The odd extension of  $\bar{u}$  satisfies the same equation in  $D_1$ . In this new coordinate system and after an odd extension in  $z_1$ , the function  $U_a$  becomes  $|z_1|^{-a}z_1$ . Thus, Proposition 6.1 is equivalent to the following proposition.

**Proposition 6.2.** *Let  $u \in C(D_1)$  be odd in  $z_1$  and even in  $z_2$  with  $\|u\|_{L^\infty(D_1)} \leq 1$  and satisfy*

$$\begin{cases} \bar{L}_a u = 0 & \text{in } D_1 \\ u = 0 & \text{on } \{z_1 = 0\}. \end{cases} \quad (6.1)$$

For any  $k \geq 0$ , there exists a polynomial  $Q$  of degree  $2k$  with  $\|Q\| \leq C$  such that  $\bar{L}_a(|z_1|^{-a}z_1Q) = 0$  in  $D_1$  and

$$|u - |z_1|^{-a}z_1Q| \leq C|z_1|^{2s}|z|^{2k+2}$$

for some constant  $C = C(a, k) > 0$ .

If  $a = 0$ , then  $u$  is harmonic and Proposition 6.2 follows easily. Instead, when  $a \neq 0$ , we prove the result from scratch in three steps. First, we construct a set homogeneous solutions of (6.1). Second, we show that these homogeneous solutions form an orthonormal basis for  $L^2(\partial D_1)$  with an appropriate weight. Third, we expand  $u|_{\partial D_1}$  in this basis, extend this expansion to the interior of  $D_1$ , and compare  $u$  to the extension.

Let  $\bar{\omega}_a(z_1, z_2) := |2z_1z_2|^a$ , and observe that  $\bar{\omega}_a$  is an  $A_2$ -Muckenhoupt weight.

**Remark 6.3** (Homogeneous Solutions). For every  $j \in \mathbb{N} \cup \{0\}$ , define the function

$$\bar{u}_j(z_1, z_2) := |z_1|^{-a}z_1\bar{Q}_j(z_1^2, z_2^2).$$

Here,  $\bar{Q}_j(z_1, z_2) := b_i z_1^i z_2^{j-i}$ ,

$$b_i := -\frac{(j-i+1)(j-i+1-s)}{i(i+s)} b_{i-1},$$

and  $b_0 = b_0(j, a)$  is chosen so that  $\|\bar{u}_j\|_{L^2(\partial D_1, \bar{\omega}_a)} = 1$ . Each  $\bar{u}_j$  is odd in  $z_1$ , even in  $z_2$ , vanishes on the  $z_2$ -axis, and satisfies

$$\bar{L}_a \bar{u}_j = 0 \quad \text{in } \mathbb{R}^2.$$

The two Green's identities below will be used in what follows. The first is applied to prove Lemma 6.5, an important estimate for the proof of Proposition 6.2. The second is utilized in Lemma 6.6 to show that the homogeneous solutions of Remark 6.3 form an orthonormal basis for  $L^2(\partial D_1, \bar{\omega}_a)$ .

**Remark 6.4** (Green's Identities). If  $u, v \in H^1(D_1, \bar{\omega}_a)$  are such that  $\bar{L}_a u = \bar{L}_a v = 0$  in  $D_1$ , then  $u$  and  $v$  satisfy the following Green's identities for  $\bar{L}_a$ :

$$\int_{D_\lambda} \nabla v \cdot \nabla u \bar{\omega}_a \, dz = \int_{\partial D_\lambda} u \partial_\nu v \bar{\omega}_a \, d\sigma \quad \forall \lambda < 1 \quad (6.2)$$

and

$$\int_{\partial D_\lambda} u \partial_\nu v \bar{\omega}_a \, d\sigma - \int_{\partial D_\lambda} v \partial_\nu u \bar{\omega}_a \, d\sigma = 0 \quad \forall \lambda < 1 \quad (6.3)$$

The following lemma shows that the (weighted)  $L^2$ -norm of  $u$  on the boundary of  $D_1$  controls the (weighted)  $L^2$ -norm of  $u$  inside  $D_1$ .

**Lemma 6.5.** *Suppose  $u \in H^1(D_1, \bar{\omega}_a)$  satisfies  $\bar{L}_a u = 0$  in  $D_1$ . Then, there exists a positive constant  $C$ , depending only on  $a$ , such that*

$$\|u\|_{L^2(D_1, \bar{\omega}_a)} \leq C \|u\|_{L^2(\partial D_1, \bar{\omega}_a)}.$$

*Proof.* Let  $\phi$  defined by

$$\phi(\lambda, u) := \int_{\partial D_\lambda} |u|^2 \bar{\omega}_a \, d\sigma,$$

where the average is taken with respect to  $\bar{\omega}_a$ . Using (6.2), we compute that  $\phi$  is increasing in  $\lambda$ . Hence,

$$\int_{D_1} |u|^2 \bar{\omega}_a \, dz \leq \phi(1, u) \int_0^1 \int_{\partial D_\lambda} \bar{\omega}_a \, d\sigma \, d\lambda = C \int_{\partial D_1} |u|^2 \bar{\omega}_a \, d\sigma,$$

as desired.  $\square$

Now, let us demonstrate that the homogeneous solutions of Remark 6.3 are an orthonormal basis for  $L^2(\partial D_1, \bar{\omega}_a)$ .

**Lemma 6.6.** *Let  $\iota_a := \|\bar{\omega}_a\|_{L^1(\partial D_1)}^{-1}$  and  $\bar{u}_j$  be as in Remark 6.3. The set  $\{\iota_a, \bar{u}_j : j = 0, 1, \dots\}$  is an orthonormal basis for  $L^2(\partial D_1, \bar{\omega}_a)$ .*

*Proof.* By construction,  $\|\iota_a\|_{L^2(\partial D_1, \bar{\omega}_a)} = \|\bar{u}_j\|_{L^2(\partial D_1, \bar{\omega}_a)} = 1$ . We show that  $\{\iota_a, \bar{u}_j : j = 0, 1, \dots\}$  is an orthogonal, dense set in  $L^2(\partial D_1, \bar{\omega}_a)$ .

We first treat the question of density. By symmetry, letting  $A := \partial D_1 \cap \{z_1, z_2 \geq 0\}$ , it suffices to show that  $\text{span}\{\iota_a, \bar{u}_j : j = 0, 1, \dots\}$  is dense in  $L^2(A, \bar{\omega}_a)$ . Furthermore, via the locally Lipschitz change of variables  $\Phi : [0, 1] \rightarrow A$  given by  $\Phi(t) := (t, \sqrt{1-t^2})$  with Jacobian  $J_\Phi(t) = (1-t^2)^{-1/2}$ , this reduces to showing that  $\text{span}\{1, \bar{w}_j : j = 0, 1, \dots\}$  is dense in  $L^2([0, 1], \bar{\mu}_a)$  where  $\bar{w}_j := \bar{u}_j \circ \Phi$  and  $\bar{\mu}_a := (\bar{\omega}_a \circ \Phi)J_\Phi$ . To this end, observe that for every  $l \in \mathbb{N} \cup \{0\}$ , there exist constants  $c_j \in \mathbb{R}$  for  $j = 0, 1, \dots, l$  such that

$$t^{2s+2l} = \sum_{j=0}^l c_j \bar{w}_j(t).$$

The Müntz-Szász theorem<sup>7</sup> implies that the family  $\text{span}\{1, t^{2s+2l} : l = 0, 1, \dots\}$  is dense in  $C([0, 1])$ . Then, since  $C([0, 1])$  is dense in  $L^2([0, 1], \bar{\mu}_a)$ , the question of density is settled.

We now show the set  $\{\iota_a, \bar{u}_j : j = 0, 1, \dots\}$  is orthogonal in  $L^2(\partial D_1, \bar{\omega}_a)$ . First, since the functions  $\bar{u}_j$  are odd in  $z_1$  and  $\bar{\omega}_a$  is even, the inner product of  $\bar{u}_j$  and  $\iota_a$  in  $L^2(\partial D_1, \bar{\omega}_a)$  is zero for every  $j \in \mathbb{N} \cup \{0\}$ . Moreover, since  $\bar{u}_j$  is homogeneous of degree  $2s + 2j$ , one computes for any  $z \in \partial D_1$ ,

$$\partial_\nu \bar{u}_j(z) = \frac{d}{d\lambda} \Big|_{\lambda=1} \bar{u}_j(\lambda z) = (2s + 2j) \bar{u}_j(z).$$

Therefore, for  $\bar{u}_k$  and  $\bar{u}_j$ , (6.3) becomes

$$0 = (2k - 2j) \int_{\partial D_1} \bar{u}_j \bar{u}_k \bar{\omega}_a \, d\sigma,$$

as desired.  $\square$

We now prove Proposition 6.2 (and thus, Proposition 6.1) using Lemmas 6.5 and 6.6 and the local boundedness property for  $\bar{L}_a$  (cf. (2.5)).

*Proof of Proposition 6.2.* By Lemma 6.6,

$$u|_{\partial D_1} = c_a \iota_a + \sum_{j=0}^{\infty} c_j \bar{u}_j|_{\partial D_1}$$

as a function in  $L^2(\partial D_1, \bar{\omega}_a)$ , and  $c_a = 0$  as  $u$  is odd in  $z_1$ .

For every  $k \in \mathbb{N} \cup \{0\}$ , let  $v_k := \sum_{j=0}^k c_j \bar{u}_j$  and note that  $\bar{L}_a v_k = 0$  in  $D_1$ . So, applying Lemma 6.5 and [16, Corollary 2.3.4] to  $w_{k+1} := u - v_k$ ,

$$\lim_{k \rightarrow \infty} \|u - v_k\|_{L^\infty(D_{1/2})} \leq C \lim_{k \rightarrow \infty} \|u - v_k\|_{L^2(D_1, \bar{\omega}_a)} \leq \lim_{k \rightarrow \infty} C \|u - v_k\|_{L^2(\partial D_1, \bar{\omega}_a)} = 0.$$

<sup>7</sup>The Müntz-Szász theorem (see [25] and [27]) says that  $\text{span}\{1, t^{p_i} : p_i > 0\}$  for  $i = 0, 1, \dots$  is dense in  $C([0, 1])$  provided that

$$\sum_{i=0}^{\infty} \frac{1}{p_i} = \infty.$$

Therefore,

$$u = \sum_{j=0}^{\infty} c_j \bar{u}_j \quad \text{in } D_{1/2}.$$

Since  $\|u\|_{L^\infty(D_1)} \leq 1$ , it follows that  $\sup_j |c_j| \leq 1$ . Consequently,

$$|w_{k+1}| \leq |u| + |v_k| \leq C$$

for a constant  $C$  depending only on  $k$  and  $a$ . For any  $0 < \lambda < 1$ , define the rescaling

$$\tilde{w}(z) := \frac{w_{k+1}(\lambda z)}{\lambda^{2s+2k+2}},$$

and observe that  $|\tilde{w}| \leq C$  in  $D_{1/2}$  thanks to the homogeneity of each term in  $w_{k+1}$ . Furthermore, the functions  $\tilde{w}$  and  $|z_1|^{-a} z_1$  vanish on  $\{z_1 = 0\}$  and satisfy  $\bar{L}_a \tilde{w} = \bar{L}_a(|z_1|^{-a} z_1) = 0$  in  $D_{1/2}$ . Applying the boundary Harnack estimate in  $D_{1/2} \cap \{z_1 > 0\}$  and recalling that  $\tilde{w}$  and  $|z_1|^{-a} z_1$  are odd in  $z_1$ , we find that

$$|\tilde{w}| \leq C |z_1|^{2s} \quad \text{in } D_{1/4}.$$

Rescaling and letting  $Q(z) := \sum_{j=0}^k c_j \bar{Q}_j(z_1^2, z_2^2)$ , we deduce that

$$|u - |z_1|^{-a} z_1 Q| \leq C |z_1|^{2s} |z|^{2k+2},$$

concluding the proof.  $\square$

With Proposition 6.1 in hand, we use a perturbative argument to prove the following, which in turn will allow us to conclude the proof of Proposition 3.3.

**Lemma 6.7.** *Suppose  $u \in C(B_1)$  is even in  $y$  with  $\|u\|_{L^\infty(D_1)} \leq 1$ , vanishes on  $\{x \leq 0, y = 0\}$ , and satisfies*

$$L_a u(z) = |y|^a \left( \frac{U_a}{r} R(x, r) + F(z) \right) \quad \text{in } D_1 \setminus \{x \leq 0, y = 0\}$$

where  $R(x, r)$  is a polynomial of degree  $k$  with  $\|R\| \leq 1$  and

$$|F(z)| \leq r^{s-1} |z|^{k+\alpha}.$$

Then, there exists a polynomial  $P(x, r)$  of degree  $k+1$  such that  $\|P\| \leq C$  and

$$|u - U_a P| \leq C U_a |z|^{k+1+\alpha}$$

for some constant  $C = C(a, k, \alpha) > 0$ .

The proof of Lemma 6.7 follows the two step improvement of flatness and iteration procedure given in the proof of Proposition 3.2, and so we omit it.<sup>8</sup>

We now prove Proposition 3.3.

*Proof of Proposition 3.3.* The equation (3.5) is invariant with respect to  $x'$ , so any partial derivative of  $u$  in an  $x'$ -direction also satisfies (3.5). Furthermore, the set  $\{x_n \leq 0, y = 0\}$ , where  $u$  vanishes, has uniformly positive  $L_a$ -capacity. It follows that solutions of (3.5) are uniformly Hölder continuous in compact subsets of  $B_1$ . In particular, as  $\|u\|_{L^\infty(B_1)} \leq 1$ , we see that for any multi-index  $\mu$ ,

$$\|D_{x'}^\mu u\|_{C^{0,\tau}(B_{1/2})} \leq C \tag{6.4}$$

for some constant  $C = C(a, |\mu|) > 0$ ; that is,  $u \in C_{x'}^\infty(B_{1/2})$ .

With the regularity of  $u$  in  $x'$  understood, we turn to understanding the regularity of  $u$  in  $z$ . To this end, notice that

$$L_a u = |y|^a \Delta_{x'} u + \operatorname{div}_z (|y|^a \nabla_z u).$$

<sup>8</sup>Since  $U_a$  and  $r$  are homogeneous when  $\Gamma$  is flat and  $U_a$  is  $a$ -harmonic away from the set  $\{x_n \leq 0, y = 0\}$ , the notion of approximating polynomials and proof are much simpler.

So, for any fixed  $x'$ ,  $u$  satisfies

$$\begin{cases} L_a u = -|y|^a \Delta_{x'} u & \text{in } D_{1,x'} \setminus \{x_n \leq 0, y = 0\} \\ u = 0 & \text{on } \{x_n \leq 0, y = 0\} \end{cases}$$

as a function of  $z$ . (Here,  $L_a$  is seen as a two-dimensional operator.) Let

$$f := -\Delta_{x'} u.$$

By (6.4),  $f$  is  $C^{0,\tau}$  in compact subsets of  $B_1$ . Up to multiplying by a constant, we may assume that  $\|f\|_{C^{0,\tau}(B_{1/2})} \leq 1$ . Then, since  $f(x', 0) = 0$  for every  $x'$ , viewed just as a function in  $z$ , we see that

$$|f(z)| \leq |z|^\tau \leq r^{s-1} |z|^\tau.$$

In particular,  $u$  satisfies the hypotheses of Lemma 6.7 with  $k = 0$  taking  $R(x_n, r) \equiv 0$  and  $F(z) = f(z)$ . Applying Lemma 6.7, we find a degree 1 polynomial  $P_0(x_n, r)$  such that

$$|u - U_a P_0| \leq C U_a |z|^{1+\tau}.$$

As  $u$  and  $f$  have the same regularity in  $z$ , it follows that

$$f(z) = U_a Q_0(x_n, r) + U_a O(|z|^{1+\tau})$$

for a polynomial  $Q_0(x_n, r)$  of degree 1. Equivalently,

$$f(z) = \frac{U_a}{r} R(x_n, r) + F(z)$$

where  $R(x_n, r) = r Q_0(x_n, r)$  is a degree 2 polynomial and, up to multiplication by a constant,  $\|R\| \leq 1$  and  $|F(z)| \leq r^{s-1} |z|^{2+\tau}$ . In this way, we can bootstrap Lemma 6.7 with  $k = 2j$  to find polynomials  $P_j(x_n, r)$  of degree  $2j + 1$  such that

$$|u - U_a P_j| \leq C U_a |z|^{2j+1+\tau}.$$

In other words, for each  $x'$ , there exists  $\phi_{x'} \in C^\infty(D_{1/2,x'})$  such that

$$u(x', z) = U_a(z) \phi_{x'}(z).$$

Since  $u$  is smooth in  $x'$  and  $U_a$  is independent of  $x'$ ,  $\phi_{x'}(z) = \Phi(x', z)$  is a smooth function of  $x'$  as well. So,

$$u(x', z) = U_a(z) \Phi(x', z).$$

This proves (3.6).

Finally, that  $U_a P$  is  $a$ -harmonic in  $B_1 \setminus \mathcal{P}$  follows by induction: decompose  $P$  into its homogeneous parts  $P(x, r) = \sum_{m=0}^k P_m(x, r)$ , where each  $P_m(x, r)$  is a homogeneous polynomial of degree  $m$ . If  $m = 0$ , then  $L_a(U_a P) = 0$  in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ . Assuming  $U_a \sum_{m=0}^l P_m(x, r)$  is  $a$ -harmonic in  $B_1 \setminus \{x_n \leq 0, y = 0\}$  for all  $l < k$ , we find that

$$v := u - U_a \sum_{m=0}^l P_m(x, r) = U_a (P_{l+1}(x, r) + o(|X|^{l+1})) \quad (6.5)$$

is  $a$ -harmonic in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ . Now, consider the rescalings of  $v$  defined by

$$\tilde{v}(X) := \frac{v(\lambda X)}{\lambda^{l+1+s}}.$$

Observe that  $L_a \tilde{v} = 0$  in  $B_1 \setminus \{x_n \leq 0, y = 0\}$  and that  $\tilde{v} \rightarrow U_a P_{l+1}$  as  $\lambda \rightarrow 0$  by (6.5). Therefore,  $L_a(U_a P_{l+1}) = 0$  in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ , which concludes the proof.  $\square$

## 7. PROOF OF THEOREM 1.1

In this section, we prove Theorem 1.1. Up to multiplication by a constant, we may assume that

$$\|\varphi\|_{C^{m,\beta}(\mathbb{R}^n)} \leq 1 \quad \text{and} \quad \|v - \varphi\|_{L^\infty(\mathbb{R}^n)} \leq 1;$$

and after a translation, rotation, and dilation, we can assume that the origin is a regular point of  $\Gamma$ , that  $\Gamma \cap B_2^*$  can be written as the graph of a  $C^{1,\sigma}$  function of  $n - 1$  variables, and that  $\partial_{\mathbb{R}^n} \mathcal{P} = \Gamma$  is locally given by

$$\Gamma = \{(x', \gamma(x'), 0) : x' \in B_1'\}$$

where  $\gamma : B_1' \rightarrow \mathbb{R}$  is such that

$$\gamma(0) = 0, \quad \nabla_{x'} \gamma(0) = 0, \quad \text{and} \quad \|\gamma\|_{C^{1,\alpha}(B_1')} \leq 1.$$

Here,  $\alpha := \min\{\beta, \sigma\}$ . Now, recalling the discussion following the statement of Theorem 1.2, our goal is to apply Proposition 5.4 and then iteratively apply Proposition 3.2 to produce a polynomial  $P$  of degree  $m - 2$  for which, after restricting to the hyperplane  $\{y = 0\}$ , we have

$$\left| \frac{\partial_i(v - \varphi)}{\partial_n(v - \varphi)} - P \right| \leq C|x|^{m-2+\alpha}.$$

The functions  $\partial_i(v - \varphi)$  and  $\partial_n(v - \varphi)$  are only defined on  $\mathbb{R}^n$ , so, to apply our higher order boundary Harnack estimates, we must first extend  $v$  and  $\varphi$  to  $\mathbb{R}^{n+1}$ . Following the notation set in the introduction, we denote the (even in  $y$ )  $a$ -harmonic extension of our solution  $v$  by  $\tilde{v}$ , which satisfies (1.2). Choosing an extension  $\tilde{\varphi}$  for the obstacle to  $\mathbb{R}^{n+1}$  is less straightforward, as our choice governed by the need for the pair  $u = \partial_i(\tilde{v} - \tilde{\varphi})$  and  $U = \partial_n(\tilde{v} - \tilde{\varphi})$  to satisfy the hypotheses of Propositions 5.4 and 3.2. The primary challenge is to show that  $U$  will satisfy the derivative estimates (5.5), (5.6), (3.3), and (3.4). With this in mind, we define

$$\tilde{\varphi}(X) := \varphi(x) + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{(-1)^j}{c_j} y^{2j} \Delta^j T^0(x) \quad (7.1)$$

where  $c_0 := 1$ ,  $c_j := 2j(2j + a - 1)c_{j-1}$ , and  $T^0 = T^0(x)$  is the  $m$ th order Taylor polynomial of  $\varphi$  at the origin. The coefficients  $c_j$  are chosen such that

$$L_a \tilde{\varphi}(X) = |y|^a \Delta(\varphi - T^0)(x).$$

Set

$$w(X) := \tilde{v}(X) - \tilde{\varphi}(X), \quad (7.2)$$

For any  $i \in 1, \dots, n$ , [8, Proposition 4.3] implies that  $\partial_i w \in C^{0,\delta}(B_1)$  for all  $0 < \delta < s$ , and, up to multiplication by universal constant,  $\|\partial_i w\|_{C^{0,\delta}(B_1)} \leq 1$ . So,  $\partial_i w$  satisfies

$$\begin{cases} L_a \partial_i w = |y|^a f_i & \text{in } B_1 \setminus \mathcal{P} \\ \partial_i w = 0 & \text{on } \mathcal{P} \end{cases} \quad \text{and} \quad f_i := \Delta(\partial_i \varphi - \partial_i T^0). \quad (7.3)$$

By construction,  $f_i = f_i(x)$  is of class  $C^{m-3,\beta}$  and

$$|f_i| \leq |x|^{m-3+\beta}. \quad (7.4)$$

Again, in order to apply the higher order boundary Harnack estimates we must justify (5.5), (5.6), (3.3), and (3.4) for  $U = \partial_n w$ . We do this in two propositions. The following addresses (3.3) and (3.4), while Proposition 7.3 below addresses (5.5) and (5.6).

**Proposition 7.1.** *Let  $\Gamma \in C^{k+2,\alpha}$  with  $0 \leq k \leq m - 3$  and  $\|\Gamma\|_{C^{k+2,\alpha}} \leq 1$ . Let  $U := \partial_n w$  for  $w$  as defined in (7.2). Let  $P_0(x, r)$  be the polynomial of degree  $k + 1$  obtained from Proposition 4.1. Then,*

$$\nabla_x U = \frac{U_a}{r} \left( s P_0 \nu + r \nabla_x P_0 + (\partial_r P_0) d\nu + O(|X|^{k+1+\alpha}) \right) \quad (7.5)$$

and

$$\nabla U \cdot \nabla r = \frac{U_a}{r} \left( s P_0 + (\partial_r P_0) r + \nabla_x P_0 \cdot (d\nu) + O(|X|^{k+1+\alpha}) \right). \quad (7.6)$$

As we shall see, to prove Proposition 7.1, we stitch together a family of analogous estimates in overlapping cones based at points on  $\Gamma$  in a neighborhood of the origin. In the cone

$$\mathcal{K} := \{|z| > |x'|\}, \quad (7.7)$$

these estimates are given by the following lemma.

**Lemma 7.2.** *Fix  $m \geq 4$  and  $0 \leq k \leq m - 3$ . Let  $\Gamma \in C^{k+2,\alpha}$  with  $\|\Gamma\|_{C^{k+2,\alpha}} \leq 1/4$ . Suppose  $U \in C(B_1)$  is even in  $y$  with  $\|U\|_{L^\infty(B_1)} \leq 1$  and*

$$\begin{cases} L_a U = |y|^a f & \text{in } B_1 \setminus \mathcal{P} \\ U = 0 & \text{on } \mathcal{P} \end{cases}$$

where  $f = f(x)$  and  $f \in C^{m-3,\alpha}(B_1)$  is such that  $\|f\|_{C^{m-3,\alpha}(B_1)} \leq 1$  with vanishing derivatives up to order  $m - 3$  at the origin. Let  $P = P_0$  be the polynomial of degree  $k + 1$  obtained in Proposition 4.1. Then,

$$|\partial_i U - \partial_i(U_a P)| \leq C \frac{U_a}{r} |X|^{k+1+\alpha} \quad \text{in } \mathcal{K} \quad (7.8)$$

and

$$|\partial_y U - \partial_y(U_a P)| \leq C |y|^{-a} |X|^{k+1+\alpha-s} \quad \text{in } \mathcal{K} \quad (7.9)$$

for some constant  $C = C(a, n, k, \alpha) > 0$ .

*Proof.* Since  $|L_a U| \leq |y|^a r^{s-1} |X|^{k+\alpha}$ , Proposition 4.1 can indeed be applied to obtain an approximating polynomial  $P(x, r)$  for  $U/U_a$  of degree  $k + 1$  with  $\|P\| \leq C$  such that

$$|U - U_a P| \leq C U_a |X|^{k+1+\alpha}.$$

For a fixed  $0 < \lambda < 1$ , define

$$\tilde{U}(X) := \frac{[U - U_a P](\lambda X)}{\lambda^{k+1+\alpha+s}}.$$

By construction,  $\|\tilde{U}\|_{L^\infty(B_1)} \leq C$ . Furthermore,

$$L_a \tilde{U} = |y|^a F \quad \text{in } B_1 \setminus \mathcal{P}_\lambda$$

where, since  $P$  is an approximating polynomial for  $U/U_a$ ,

$$F(X) := \frac{f(\lambda x)}{\lambda^{k-1+\alpha+s}} - \frac{U_{a,\lambda}}{r_\lambda} \frac{h(\lambda X)}{\lambda^{k+\alpha}}.$$

Here,  $h(X) = \sum_{l=0}^k r^l h_l(x)$ , and  $h_l \in C^{k,\alpha}(B_1^*)$  have vanishing derivatives up to order  $k - l$  at zero; recall the discussion on approximating polynomials in Section 4. We decompose  $\tilde{U}$  as  $\tilde{U} = \tilde{U}_1 - \tilde{U}_2$  where  $\tilde{U}_i \equiv 0$  on  $\mathcal{P}_\lambda$  and  $L_a \tilde{U}_i = |y|^a F_i$  with

$$F_1(x) := \frac{f(\lambda x)}{\lambda^{k-1+\alpha+s}} \quad \text{and} \quad F_2(X) := \frac{U_{a,\lambda}}{r_\lambda} \frac{h(\lambda X)}{\lambda^{k+\alpha}}.$$

Notice that  $F_1 = F_1(x)$  is of class  $C^{1,\alpha}$  with  $C^{1,\alpha}$ -norm bounded independently of  $\lambda$  and that  $F_2 \equiv 0$  on  $\mathcal{P}_\lambda$ .

*Proof of (7.9).* Since  $\tilde{U}$  vanishes on  $\mathcal{P}_\lambda$ , by (2.3),

$$||y|^a \partial_y F(X)| \leq C \frac{(r_\lambda - d_\lambda)^{1-s}}{r_\lambda^2} |X|^{k+\alpha} + C |y|^{a+1} \frac{U_{a,\lambda}}{r_\lambda^3} |X|^{k+\alpha} \leq C$$

in  $\mathcal{C} \cap (B_{7/8} \setminus \overline{B}_{1/8})$  where  $\mathcal{C} := \{2|z| > |x'|\}$ . Hence, Proposition 2.3 and Corollaries 2.4 and 2.5 (applied to  $\tilde{U}_2$  and  $\tilde{U}_1$  respectively) imply that

$$||y|^a \partial_y \tilde{U}| \leq C \quad \text{in } \mathcal{K} \cap (B_{3/4} \setminus \overline{B}_{1/4}).$$

Expressing this in terms of  $U$  and rescaling, (7.9) follows as  $\lambda > 0$  was arbitrary.



*Proof of (7.8).* Notice that  $F$  is bounded independently of  $\lambda$  in the region  $\mathcal{C} \cap (B_{7/8} \setminus \overline{B}_{1/8})$ . Since  $\tilde{U}$  vanishes on  $\mathcal{P}_\lambda$ , Proposition 2.3 and Corollary 2.4 imply that

$$|\partial_i \tilde{U}| \leq C \quad \text{in } \mathcal{K} \cap (B_{3/4} \setminus \overline{B}_{1/4}).$$

We need to improve this inequality to

$$|\partial_i \tilde{U}| \leq CU_{a,\lambda} \quad \text{in } \mathcal{K} \cap (B_{3/4} \setminus \overline{B}_{1/4}). \quad (7.10)$$

Let  $\mathcal{K}^\pm$  be the upper and lower halves of  $\mathcal{K}$  with respect to  $x_n$ , that is,

$$\mathcal{K}^+ := \mathcal{K} \cap \{x_n \geq 0\} \quad \text{and} \quad \mathcal{K}^- := \mathcal{K} \cap \{x_n < 0\}. \quad (7.11)$$

In  $\mathcal{K}^+ \cap (B_{3/4} \setminus \overline{B}_{1/4})$ , we see that  $U_{a,\lambda} \geq c$ , and so (7.10) is immediate in this region. On the other hand, by Corollaries 2.6 and 2.7 (applied to  $\tilde{U}_1$  and  $\tilde{U}_2$  respectively), (7.10) holds in  $\mathcal{K}^- \cap (B_{3/4} \setminus \overline{B}_{1/4})$ . Thus, (7.8) follows as  $\lambda > 0$  was arbitrary.  $\square$

We now prove Proposition 7.1. The idea is to define a different extension  $\tilde{\varphi}^{x_0}$  of  $\varphi$  at every point in  $\Gamma \cap B_{1/2}^*$  in such a way that allows us to apply Lemma 7.2 to  $\partial_n(\tilde{v} - \tilde{\varphi}^{x_0})$  in cones based at  $x_0$ . Then, we patch the estimates from Lemma 7.2 together and conclude.

*Proof of Proposition 7.1.* For each  $x_0 \in \Gamma \cap B_{1/2}^*$ , define

$$\tilde{\varphi}^{x_0}(X) := \varphi(x) + \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{(-1)^j}{c_j} y^{2j} \Delta^j T^{x_0}(x),$$

with  $c_j$  as in (7.1) and  $T^{x_0}(x)$  the  $m$ th order Taylor polynomial of  $\varphi$  at  $x_0$ . Set

$$w^{x_0}(X) := \tilde{v}(X) - \tilde{\varphi}^{x_0}(X).$$

Letting  $X_0 := (x_0, 0) \in \mathbb{R}^{n+1}$ , we again see that  $\|\partial_n w^{x_0}\|_{C^{0,\delta}(B_1(X_0))} \leq 1$  and

$$\begin{cases} L_a \partial_n w^{x_0} = |y|^a f_n^{x_0} & \text{in } B_1(X_0) \setminus \mathcal{P} \\ \partial_n w^{x_0} = 0 & \text{on } \mathcal{P} \end{cases} \quad \text{where} \quad f_n^{x_0} := \Delta(\partial_n \varphi - \partial_n T^{x_0});$$

and, by construction,  $f_n^{x_0} = f_n^{x_0}(x)$  is of class  $C^{m-3,\beta}$  and  $|f_n^{x_0}| \leq |x - x_0|^{m-3+\beta}$ . So, up to a dilation, we apply Lemma 7.2 to  $U = U^{X_0} := \partial_n w^{x_0}$  for every  $x_0 \in \Gamma \cap B_{1/2}^*$  with right-hand side  $f = f_n^{x_0}$ . As  $\Gamma \in C^{k+2,\alpha}$ , after Taylor expanding  $\nu_i$  and  $d$ , we have

$$\left| \partial_i U^{X_0} - \frac{U_a}{r} P_{X_0}^i \right| \leq \frac{U_a}{r} C |X - X_0|^{k+1+\alpha} \quad \text{in } \mathcal{K}_{X_0} \quad (7.12)$$

and

$$\left| \nabla U^{X_0} \cdot \nabla r - \frac{U_a}{r} P_{X_0}^r \right| \leq \frac{U_a}{r} C |X - X_0|^{k+1+\alpha} \quad \text{in } \mathcal{K}_{X_0} \quad (7.13)$$

where  $P_{X_0}^i$  and  $P_{X_0}^r$  are polynomials of degree  $k+1$  and  $\mathcal{K}_{X_0}$  is the rotation and translation of  $\mathcal{K}$  centered at  $x_0$  pointing in the direction  $\nu(x_0)$ .

We now show that (7.12) and (7.13) hold for  $U^0$  in all of  $B_1$ . Given any  $X \in B_1$ , let  $X_0 \in \Gamma$  be such that  $r(X) = |X - X_0|$ ; note that  $X \in \mathcal{K}_{X_0}$  and

$$|X - X_0| \leq |X|. \quad (7.14)$$

Then,

$$\left| \partial_i U^0 - \frac{U_a}{r} P_0^i \right| \leq \left| \partial_i U^{X_0} - \frac{U_a}{r} P_{X_0}^i \right| + \left| \partial_i U^0 - \partial_i U^{X_0} \right| + \frac{U_a}{r} |P_{X_0}^i - P_0^i| = \text{I} + \text{II} + \text{III}.$$

As  $X \in \mathcal{K}_{X_0}$ , (7.12) and (7.14) imply that

$$\text{I} \leq C \frac{U_a}{r} |X|^{k+1+\alpha}.$$

Note that

$$\text{II} \leq \sum_{j=1}^{\lfloor \frac{m}{2} \rfloor + 1} \frac{(-1)^j}{c_j} y^{2j} |\Delta^j \partial_{in}(T^{x_0} - T^0)|.$$

If  $2j + 2 > m$ , then  $\Delta^j \partial_{in} T^0 = \Delta^j \partial_{in} T^{x_0} = 0$ . On the other hand, if  $2j + 2 \leq m$ , then  $\Delta^j \partial_{in} T^0$  and  $\Delta^j \partial_{in} T^{x_0}$  are the Taylor polynomials of degree  $m - 2j - 2$  at zero and  $x_0$  respectively for  $\Delta^j \partial_{in} \varphi$ . Therefore, again using (7.14),

$$|\Delta^j \partial_{in}(T^{x_0} - T^0)| \leq C|X|^{m-2j-2+\alpha}.$$

Recalling (2.3) and that  $m - 2 \geq k + 1$ , we see that

$$\text{II} \leq C|y|^{2j} |\Delta^j \partial_{in}(T^{x_0} - T^0)| \leq C \frac{U_a}{r} |X|^{m-2+\alpha} \leq C \frac{U_a}{r} |X|^{k+1+\alpha}.$$

Finally, to bound III, let  $\lambda := |X_0|$ . Since  $\Gamma \subset \{|z| < |x'|\}$ , the ball  $B_{\lambda/2}(2\lambda e_n)$  is contained in  $\mathcal{K} \cap \mathcal{K}_{X_0}$ . Observe that

$$\frac{U_a}{r} |P_{X_0}^i - P_0^i| \leq \left| \partial_i U^{X_0} - \frac{U_a}{r} P_{X_0}^i \right| + \left| \partial_i U^0 - \frac{U_a}{r} P_0^i \right| + |\partial_i U^{X_0} - \partial_i U^0| \quad \text{in } B_{\lambda/2}(2\lambda e_n).$$

Noting that  $|Z|$  and  $|Z - X_0|$  are of order  $\lambda$  for all  $Z \in B_{\lambda/2}(2\lambda e_n)$ , the bound on II and (7.12) imply that  $\|P_{X_0}^i - P_0^i\|_{L^\infty(B_{\lambda/2}(2\lambda e_n))} \leq C\lambda^{k+1+\alpha}$ . Hence,

$$\|P_{X_0}^i - P_0^i\|_{L^\infty(B_{4\lambda})} \leq C\lambda^{k+1+\alpha},$$

and, in particular, we determine that

$$\text{III} \leq C \frac{U_a}{r} |X|^{k+1+\alpha}.$$

We conclude that (7.5) holds for  $U^0$  in  $B_1$ .

An identical argument shows that (7.6) holds for  $U^0$  in  $B_1$ .  $\square$

Now we address the case when  $\Gamma \in C^{1,\alpha}$ . The following proposition shows that (5.5) and (5.6) hold for  $U = \partial_n w$ , with  $w$  defined as in (7.2).

**Proposition 7.3.** *Let  $\Gamma \in C^{1,\alpha}$  with  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ . Let  $U \in C(B_1)$  be even in  $y$  and normalized so that  $U(e_n/2) = 1$ . Let  $U \equiv 0$  on  $\mathcal{P}$  and  $U > 0$  in  $B_1 \setminus \mathcal{P}$ , and suppose  $U$  satisfies*

$$L_a U = |y|^\alpha f \quad \text{in } B_1 \setminus \mathcal{P}$$

where  $f = f(x)$  and  $f \in C^{0,1}(B_1)$  with  $\|f\|_{C^{0,1}(B_1)} \leq 1$ . Let  $p'$  be the constant obtained in Proposition 5.2. Then,

$$|\nabla_x U - p' \nabla_x U_a| \leq C \frac{U_a}{r} |X|^\alpha$$

and

$$|\partial_y U - p' \partial_y U_a| \leq C |y|^{-a} r^{-s} |X|^\alpha$$

for some constant  $C = C(a, n, \alpha) > 0$ .

*Proof.* Note that Proposition 5.2 can be applied because  $|L_a U| \leq |y|^a r^{s-2+\alpha}$ . Let  $Z$  be a point of differentiability for  $r$  with distance  $\lambda/2$  from  $\Gamma$ . Up to a translation, we may assume that the closest point on  $\Gamma$  to  $Z$  is the origin. So, at  $Z$ , we have that

$$U_a = \bar{U}_a, \quad r = \lambda/2, \quad \text{and} \quad \nabla U_a = \nabla \bar{U}_a. \quad (7.15)$$

Set

$$\tilde{U}(X) := \frac{[U - p' \bar{U}_a](\lambda X)}{\lambda^{\alpha+s}}$$

and  $\mathcal{C} := \{2|z| > |x'|\}$ , and let  $\mathcal{K}$  and  $\mathcal{K}^\pm$  be as in (7.7) and (7.11). Arguing as in the proof of Lemma 5.1, where we obtained that  $|U_{a,*} - U_a| \leq C U_a r^\alpha$ , we see that

$$|U - p' \bar{U}_a| \leq |U - p' U_a| + |p'| |\bar{U}_a - U_a| \leq C \lambda^{\alpha+s} \quad \text{in } \mathcal{C} \cap \{\lambda/8 \leq |z| \leq 7\lambda/8\}.$$

Thus,  $|\tilde{U}| \leq C$  in  $\mathcal{C} \cap \{\lambda/8 \leq |z| \leq 7\lambda/8\}$  for all  $\lambda > 0$ . Notice that

$$L_a \tilde{U} = |y|^a F \quad \text{in } B_1 \setminus \mathcal{P}_\lambda$$

where, recalling that  $\tilde{U}_a$  is  $a$ -harmonic in  $B_1 \setminus \{x_n \leq 0, y = 0\}$ ,

$$F(X) = \lambda^{2-s-\alpha} f(\lambda x).$$

Observe that  $F = F(x)$ ,  $F \in C^{0,1}(B_1)$ , and  $\|F\|_{C^{0,1}(B_1)} \leq 1$ . Arguing as in Lemma 7.2 and by (7.15), we find that

$$|\partial_y U - p' \partial_y U_a| \leq Cr^{\alpha-s} |y|^{-a} \quad \text{and} \quad |\nabla_x U - p' \nabla_x U_a| \leq CU_a r^{\alpha-1} \quad \text{at } Z.$$

Moreover, since the origin was distinguished by an arbitrary translation,

$$|\partial_y U - p'_{X_0} \partial_y U_a| \leq Cr^{\alpha-s} |y|^{-a} \quad \text{and} \quad |\nabla_x U - p'_{X_0} \nabla_x U_a| \leq CU_a r^{\alpha-1}$$

for every  $X \in B_{1/2}$  at which  $r$  is differentiable, letting  $X_0$  be the projection of  $X$  onto  $\Gamma$  and  $p'_{X_0}$  be the constant corresponding to the expansion of  $U$  at  $X_0$ . Since

$$|p'_{X_0} - p'| \leq C|X|^\alpha,$$

the lemma follows.  $\square$

Finally, we prove Theorem 1.1.

*Proof of Theorem 1.1.* Let  $U = \partial_n w$  and  $u = \partial_i w$ , with  $w$  as defined in (7.2). Thanks to the rescalings at the beginning of this section, we have  $\|\Gamma\|_{C^{1,\alpha}} \leq 1$ . Up to possible further rescaling,  $U := \partial_n w > 0$  in  $B_1$ ; see [8]. Thanks to (7.3), (7.4), and Proposition 7.3, the remaining hypotheses of Proposition 5.4 are satisfied up to multiplication by a universal constant. So, applying Proposition 5.4 to  $u$  and  $U$ , we obtain the existence of a polynomial  $P$  of degree 1 that, after a Taylor expansion of  $d$ , yields

$$\left| \frac{\partial_i(v - \varphi)}{\partial_n(v - \varphi)} - P \right| \leq C|x|^{1+\alpha}.$$

Up to translation and rotation, we may argue identically with  $x_0 \in \Gamma \cap B_{1/2}^*$  in place of the origin. Hence,  $\partial_i(v - \varphi)/\partial_n(v - \varphi) \in C^{1,\alpha}(B_{1/2}^*)$ . By a well-known argument (cf. [26, Theorem 6.9]), this implies that  $\Gamma \cap B_{1/4}^* \in C^{2,\alpha}$

Passing from  $C^{k+2,\alpha}$  to  $C^{k+3,\alpha}$  for  $0 \leq k \leq m-4$  is identical; here Proposition 7.1 is used to show the hypotheses (3.3) and (3.4) of Proposition 3.2 are satisfied. Applying Proposition 3.2, we find that there exists a polynomial  $P$  of degree  $k+2$  such that, after restricting to the hyperplane  $\{y=0\}$ ,

$$\left| \frac{\partial_i(v - \varphi)}{\partial_n(v - \varphi)} - P \right| \leq C|x|^{k+2+\alpha}.$$

In turn, this implies that  $\Gamma \in C^{k+3,\alpha}$ . Arguing iteratively for  $k = 0, \dots, m-4$ , the theorem is proved.  $\square$

## 8. APPENDIX

In this section, we prove Lemmas 5.1 and 5.3.

*Proof of Lemma 5.1.* The regularizations of  $r$  and  $U_a$  are constructed in the same way as the analogous regularizations in [14]. First, we smooth the signed distance function  $d$  via convolution in  $\lambda$ -neighborhoods of  $\Gamma$ . Then, we define approximations  $r_\lambda$  and  $U_{a,\lambda}$  in geometrically shrinking annuli, and we patch them together in a smooth way. The functions  $r_\lambda$  and  $U_{a,\lambda}$  here should not be confused with the rescalings of  $r$  and  $U_a$  defined in (3.13).

The functions  $d, r$ , and  $U_a$  are locally Lipschitz in  $B_1 \setminus \mathcal{P}$ , and are therefore differentiable almost everywhere. When we speak of their derivatives, we assume we are at a point of differentiability.

*Step 1: Construction and estimates for the function  $d_\lambda$ .*

Define the set  $\mathcal{D}_\lambda := \{x \in \mathbb{R}^n : |d| < 4\lambda\}$ . Let  $\eta \in C_0^\infty(B_{1/50}^*)$  be a positive, radially symmetric function that integrates to 1, and set

$$d_\lambda := d * \eta_\lambda$$

where  $\eta_\lambda := \lambda^{-n} \eta(x/\lambda)$ . As in [14], the following estimates hold for  $d_\lambda$  in  $\mathcal{D}_\lambda$ :

$$|d_\lambda - d| \leq C\lambda^{1+\alpha}, \quad |\nabla d_\lambda - \nabla d| \leq C\lambda^\alpha, \quad \text{and} \quad |D^2 d_\lambda| \leq C\lambda^{\alpha-1}. \quad (8.1)$$

In particular, the gradient estimate implies

$$|\nabla d_\lambda| = 1 + O(\lambda^\alpha). \quad (8.2)$$

*Step 2: Construction and estimates for the function  $r_\lambda$ .*

Let

$$\mathcal{R}_\lambda := \{\lambda/2 < r < 4\lambda\} \subset \mathcal{D}_\lambda \quad \text{and} \quad r_\lambda := (d_\lambda^2 + y^2)^{1/2} \quad \text{in } \mathcal{R}_\lambda. \quad (8.3)$$

The following estimates hold in  $\mathcal{R}_\lambda$ :

$$|r_\lambda - r| \leq C\lambda^{1+\alpha}, \quad |\nabla r_\lambda - \nabla r| \leq C\lambda^\alpha, \quad |D^2 r_\lambda| \leq C\lambda^{-1}, \quad \text{and} \quad \left| \Delta r_\lambda - \frac{1}{r} \right| \leq C\lambda^{\alpha-1}. \quad (8.4)$$

Consequently, we have that

$$\left| \frac{r_\lambda}{r} - 1 \right| \leq C\lambda^\alpha, \quad \left| \frac{1}{r_\lambda} - \frac{1}{r} \right| \leq C\lambda^{\alpha-1}, \quad (8.5)$$

and

$$|\nabla r_\lambda| = 1 + O(\lambda^\alpha). \quad (8.6)$$

All of these estimates were shown in [14], so we do not reprove them here. Furthermore, we find that

$$\left| L_a r_\lambda - \frac{2(1-s)|y|^a}{r} \right| \leq C|y|^a \lambda^{\alpha-1}. \quad (8.7)$$

To show (8.7), we express  $L_a(r_\lambda^2)$  in two different ways:

$$2r_\lambda L_a r_\lambda + 2|y|^a |\nabla r_\lambda|^2 = L_a(r_\lambda^2) = 2|y|^a d_\lambda \Delta_x d_\lambda + 2|y|^a |\nabla d_\lambda|^2 + 4(1-s)|y|^a.$$

Then, (8.6), (8.2), and the third bound in (8.1) imply that

$$r_\lambda L_a r_\lambda = |y|^a (-|\nabla r_\lambda|^2 + d_\lambda \Delta_x d_\lambda + |\nabla d_\lambda|^2 + 2(1-s)) = |y|^a (2(1-s) + O(\lambda^\alpha)).$$

Hence, (8.7) follows from (8.5).

*Step 3: Construction and estimates for the function  $U_{a,\lambda}$ .*

We define

$$U_{a,\lambda} := \left( \frac{d_\lambda + r_\lambda}{2} \right)^s \quad \text{in } \mathcal{R}_\lambda,$$

with  $\mathcal{R}_\lambda$  defined as in (8.3). The following bounds hold for  $U_{a,\lambda}$  in  $\mathcal{R}_\lambda$ :

$$\left| \frac{U_{a,\lambda}}{U_a} - 1 \right| \leq C\lambda^a, \quad |\nabla_x U_{a,\lambda} - \nabla_x U_a| \leq C\lambda^{\alpha-1+s}, \quad \text{and} \quad |\partial_y U_{a,\lambda} - \partial_y U_a| \leq \frac{C\lambda^{\alpha+s}}{|y|}, \quad (8.8)$$

as well as

$$|L_a U_{a,\lambda}| \leq C|y|^a \lambda^{\alpha-2+s}. \quad (8.9)$$

These estimates must be shown separately in the regions

$$\mathcal{R}_\lambda^+ := \mathcal{R}_\lambda \cap \left\{ d \geq -\frac{r}{2} \right\} \quad \text{and} \quad \mathcal{R}_\lambda^- := \mathcal{R}_\lambda \cap \left\{ d < -\frac{r}{2} \right\}.$$

*Step 3a: Estimates in  $\mathcal{R}_\lambda^+$ .* In  $\mathcal{R}_\lambda^+$ , the functions  $U_a$  and  $U_{a,\lambda}$  are comparable to  $\lambda^s$ . Also,

$$U_{a,\lambda} = U_a \left( \frac{r_\lambda + d_\lambda}{r + d} \right)^s.$$

From the established bounds on  $r_\lambda$  and  $d_\lambda$  ((8.1) and (8.4)), we have

$$\frac{r_\lambda + d_\lambda}{r + d} = 1 + O(\lambda^\alpha) \quad \text{and} \quad \nabla \left( \frac{r_\lambda + d_\lambda}{r + d} \right) = O(\lambda^{\alpha-1}).$$

Therefore,

$$\left| \frac{U_{a,\lambda}}{U_a} - 1 \right| = \left| \left( \frac{r_\lambda + d_\lambda}{r + d} \right)^s - 1 \right| = O(\lambda^\alpha) \quad \text{and} \quad \nabla U_{a,\lambda} = \nabla U_a + O(\lambda^{s-1+\alpha}),$$

proving all three estimates in (8.8) in  $\mathcal{R}_\lambda^+$ . (Notice that we have actually shown a stronger estimate for the  $y$ -derivative in  $\mathcal{R}_\lambda^+$ .) To determine the bound on  $L_a U_{a,\lambda}$ , we compute

$$sL_a(U_{a,\lambda}^{1/s}) = U_{a,\lambda}^{1/s-1} L_a U_{a,\lambda} + |y|^a \left( \frac{1-s}{s} \right) U_{a,\lambda}^{1/s-2} |\nabla U_{a,\lambda}|^2.$$

On the other hand,

$$L_a(U_{a,\lambda}^{1/s}) = L_a \left( \frac{r_\lambda + d_\lambda}{2} \right) = \frac{(1-s)|y|^a}{r} + |y|^a O(\lambda^{\alpha-1}).$$

Together these estimates imply that

$$U_{a,\lambda}^{1/s-1} L_a U_{a,\lambda} = s|y|^a \left( - \left( \frac{1-s}{s^2} \right) U_{a,\lambda}^{1/s-2} |\nabla U_{a,\lambda}|^2 + \frac{2(1-s)}{r} + O(\lambda^{\alpha-1}) \right) = |y|^a O(\lambda^{\alpha-1}),$$

where the second equality follows by (8.5) and using that

$$|\nabla U_{a,\lambda}|^2 = s^2 \frac{U_{a,\lambda}^{2-1/s}}{r_\lambda} + O(\lambda^{\alpha-2+2s}) \quad \text{in } \mathcal{R}_\lambda^+.$$

Multiplying by  $U_{a,\lambda}^{1-1/s}$ , we obtain (8.9).

*Step 3b: Estimates in  $\mathcal{R}_\lambda^-$ .* In  $\mathcal{R}_\lambda^-$ , the functions  $U_a$  and  $U_{a,\lambda}$  are comparable to  $|y|^{2s}\lambda^{-s}$ . Indeed,  $\frac{3}{2}r < r-d < 2r$  and  $r+d = y^2/(r-d)$ . Thus, from (8.1) and (8.4), we observe that

$$\frac{r-d}{r_\lambda - d_\lambda} = 1 + O(\lambda^\alpha) \quad \text{and} \quad \nabla \left( \frac{r-d}{r_\lambda - d_\lambda} \right) = O(\lambda^{\alpha-1}).$$

As a consequence,

$$U_{a,\lambda} = U_a \left( \frac{r-d}{r_\lambda - d_\lambda} \right)^s = U_a (1 + O(\lambda^\alpha)).$$

To see the  $x$ -gradient estimate in (8.8), since  $|\nabla_x U_a| = sU_a/r \leq C\lambda^{s-1}$ , we compute

$$\nabla_x U_{a,\lambda} = \nabla_x U_a + O(\lambda^{\alpha-1+s}).$$

Similarly, using that  $|\partial_y U_a| = s|y|^{-1}U_a(r-d)/r \leq C|y|^{-1}\lambda^s$ , we find that

$$\partial_y U_{a,\lambda} = \partial_y U_a (1 + O(\lambda^\alpha)) + U_a O(\lambda^{\alpha-1}) = \partial_y U_a + |y|^{-1} O(\lambda^{\alpha+s}).$$

This proves (8.8). Finally, we compute  $L_a U_{a,\lambda}$  directly:

$$\begin{aligned} 2^s L_a U_{a,\lambda} &= L_a (|y|^{2s} (r_\lambda - d_\lambda)^{-s}) = (1+2s) \frac{y}{|y|} \partial_y ((r_\lambda - d_\lambda)^{-s}) + |y| \Delta ((r_\lambda - d_\lambda)^{-s}) \\ &= -\frac{s(1+2s)|y|}{r_\lambda (r_\lambda - d_\lambda)^{s+1}} - \frac{s|y| \Delta (r_\lambda - d_\lambda)}{(r_\lambda - d_\lambda)^{s+1}} + \frac{s(s+1)|y| |\nabla (r_\lambda - d_\lambda)|^2}{(r_\lambda - d_\lambda)^{s+2}}. \end{aligned}$$

Noting that  $|\nabla (r_\lambda - d_\lambda)|^2 = r_\lambda^{-2} (2r_\lambda (r_\lambda - d_\lambda) + O(\lambda^{2+\alpha}))$ , recalling (8.4) and (8.5), and simplifying, we see that (8.9) holds.

*Step 4: Construction and estimates for  $r_*$  and  $U_{a,*}$ .*

The functions  $r_*$  and  $U_{a,*}$  are constructed by letting  $\lambda_k = 4^{-k}$  and smoothly interpolating between  $r_{\lambda_k}$  and  $U_{a,\lambda}$  respectively with

$$r_* := \begin{cases} r_{\lambda_k} & \text{in } \mathcal{R}_{\lambda_k} \cap \{r \leq 2\lambda_k\} \\ r_{4\lambda_k} & \text{in } \mathcal{R}_{\lambda_k} \cap \{r > 3\lambda_k\} \end{cases} \quad \text{and} \quad U_{a,*} := \begin{cases} U_{a,\lambda_k} & \text{in } \mathcal{R}_{\lambda_k} \cap \{r \leq 2\lambda_k\} \\ U_{a,4\lambda_k} & \text{in } \mathcal{R}_{\lambda_k} \cap \{r > 3\lambda_k\} \end{cases}$$

with  $r_*$  and  $U_{a,*}$  smooth in the intermediate region. More specifically, this is accomplished by defining

$$\psi(t) := \begin{cases} 1 & \text{if } t \leq 2 + \frac{1}{4} \\ 0 & \text{if } t \geq 2 + \frac{3}{4} \end{cases}$$

that is smooth for  $2 + \frac{1}{4} < t < 2 + \frac{3}{4}$ , letting  $\Psi := \psi(r_\lambda/\lambda)$ , and setting

$$r_* := \Psi r_\lambda + (1 - \Psi)r_{4\lambda} \quad \text{and} \quad U_{a,*} := \Psi U_{a,\lambda} + (1 - \Psi)U_{a,4\lambda}.$$

We use the estimates in (8.4) to show the following bounds on  $r_*$  in  $\mathcal{R}_\lambda$ :

$$\begin{aligned} |r_* - r| &\leq C\lambda^{1+\alpha}, & |\nabla r_* - \nabla r| &\leq C\lambda^\alpha, \\ |\partial_y r_* - \partial_y r| &\leq C|y|^a r^{s-1} U_a \lambda^\alpha, & \text{and} & \left| L_a r_* - \frac{2(1-s)|y|^a}{r} \right| &\leq C|y|^a \lambda^{\alpha-1}. \end{aligned} \quad (8.10)$$

Indeed, the first estimate follows from (8.4) and since  $0 \leq \Psi \leq 1$ . Next, keeping (8.6) in mind, the following estimates hold for  $\Psi$ :

$$|\nabla \Psi| \leq C\lambda^{-1}, \quad |D^2 \Psi| \leq C\lambda^{-2}, \quad \text{and} \quad |\partial_y \Psi| \leq C|y|\lambda^{-2}. \quad (8.11)$$

The remaining three inequalities in (8.10) follow from (8.11), the established estimates on  $r_\lambda$ , and an explicit computation.

Next using (8.8) and (8.9), we show the following hold for  $U_{a,*}$  in  $\mathcal{R}_\lambda$ :

$$\left| \frac{U_{a,*}}{U_a} - 1 \right| \leq C\lambda^\alpha, \quad \left| \frac{|\nabla U_{a,*}|}{|\nabla U_a|} - 1 \right| \leq C\lambda^\alpha, \quad \text{and} \quad |L_a U_{a,*}| \leq C|y|^a \lambda^{\alpha-2+s}. \quad (8.12)$$

The first inequality follows trivially. Since  $U_a/r \leq C|\nabla U_a|$ , we find that  $\nabla U_{a,\lambda} = \nabla U_a(1 + O(\lambda^\alpha))$ . Hence, the second inequality in (8.12) holds utilizing the first inequalities in (8.8) and (8.11). Finally, from (8.8) and (8.11) and an simple computation, one justifies the third estimate in (8.12).

Given (8.10) and (8.12), the lemma follows.  $\square$

We now prove Lemma 5.3.

*Proof of Lemma 5.3.* We construct upper and lower barriers. Define the upper barrier

$$v_+ := U_{a,*} - U_{a,*}^\beta$$

for some  $\beta > 1$  that will be chosen. Note that  $v_+ \geq 0$  on  $\partial B_1 \cup \mathcal{P}$  since  $\beta > 1$ . From Lemma 5.1, we have that

$$|L_a U_{a,*}| \leq C_* \varepsilon |y|^a r^{\alpha-2+s}, \quad |U_{a,*}| \leq C_* r^s, \quad \text{and} \quad |\nabla U_{a,*}|^2 \geq c \frac{U_{a,*}^{2-1/s}}{r}.$$

The third inequality holds provided that  $0 < \varepsilon \leq 1/2C_*$ . Indeed,

$$|\nabla U_{a,*}| \geq (1 - C_* \varepsilon) |\nabla U_a| = (1 - C_* \varepsilon) s \frac{U_a^{1-1/2s}}{r^{1/2}} \geq c \frac{U_{a,*}^{1-1/2s}}{r^{1/2}}.$$

Now, observe that

$$\begin{aligned} L_a v_+ &= L_a U_{a,*} - \beta U_{a,*}^{\beta-1} L_a U_{a,*} - |y|^a \beta(\beta-1) U_{a,*}^{\beta-2} |\nabla U_{a,*}|^2 \\ &\leq |y|^a (C_* \varepsilon r^{\alpha-2+s} - c r^{s\beta-2}) \end{aligned}$$

if  $\beta - 1/s < 0$ . Setting  $\beta = 1 + \alpha/s$ , we see that  $\beta - 1/s < 0$  (recall that  $\alpha \in (0, 1-s)$ ), and so choosing  $\varepsilon > 0$  smaller if necessary, we deduce that

$$L_a v_+ \leq -c|y|^a r^{\alpha-2+s}.$$

We take  $v_- := -v_+$  as a lower barrier. The maximum principle ensures that

$$|u| \leq C|v_{\pm}| \leq CU_{a,*} \leq CU_a.$$

□

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