ON THE (IN)STABILITY OF THE IDENTITY MAP IN OPTIMAL TRANSPORTATION

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Abstract. We collect some examples of optimal transports in order to explore the (in)stability of the identity map as an optimal transport. First, we consider density and domain perturbations near regular portions of domains. Second, we investigate density and domains deformations to non-regular parts of domains. Here, we restrict our attention to two dimensions and focus near 90 degree corners.

1. Introduction

The optimal transport problem for quadratic cost asks whether or not it is possible to find a map that minimizes the total cost of moving a distribution of mass μ to another ν given the cost of moving x to y is measured by the squared distance between x and y; concisely written, it is

$$\min\left\{\int |x-T(x)|^2\,\mathrm{d}\mu(x) : T_{\#}\mu = \nu\right\}.$$

Under certain conditions on μ and ν, the existence of a unique (μ-a.e.) minimizing map, an optimal transport, was first discovered by Brenier in [1] — he characterized optimal transports as gradients of convex functions. The regularity of optimal maps is a delicate question, guaranteed only under natural but strong geometric conditions.

Let $\mu = f(x)\,\mathrm{d}x$ and $\nu = g(y)\,\mathrm{d}y$, and set $X = \{f > 0\}$ and $Y = \{g > 0\}$, which we assume to be open, bounded subsets of $\mathbb{R}^n$. If f and g are bounded away from zero and infinity on X and Y respectively and Y is convex, then Caffarelli showed, in [2], that u is a strictly convex (Alexandrov) solution to the Monge-Ampère equation

$$\det(D^2u) = \frac{f}{g \circ \nabla u} \text{ in } X.$$

From here, he developed a regularity theory for mappings with convex potentials, part of which we now recall. Under the assumption that Y is convex ([2]):

- If $\lambda \leq f, g \leq 1/\lambda$ with $\lambda > 0$, then $\nabla u \in C^{0,\sigma}_{\text{loc}}(X)$ for some $\sigma \in (0,1)$.
- If, in addition, $f \in C^{k,\alpha}_{\text{loc}}(X)$ and $g \in C^{k,\alpha}_{\text{loc}}(Y)$, then $\nabla u \in C^{k+1,\alpha}_{\text{loc}}(X)$, for $k \geq 0$ and $\alpha \in (0,1)$.

Under the assumption that both X and Y are convex ([3]):

- If $\lambda \leq f, g \leq 1/\lambda$ with $\lambda > 0$, then $\nabla u \in C^{0,\sigma}(\overline{X})$ for some $\sigma \in (0,1)$.

Under the assumption that both X and Y are smooth and uniformly convex ([4]):

- If $f \in C^{k,\alpha}(\overline{X})$ and $g \in C^{k,\alpha}(\overline{Y})$ with $f, g > 0$, then $\nabla u \in C^{k+1,\alpha}(\overline{X})$, for $k \geq 0$ and $\alpha \in (0,1)$.

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That said, given any set E, the optimal transport taking (the constant density 1 on) E to (the constant density 1 on) E is the identity map.

In general, when we say the optimal transport taking a set X to a set Y, we mean the optimal transport taking the density $\mathbf{1}_X$ to the density $\mathbf{1}_Y$ (necessarily, $|X| = |Y|$).

In this paper, we study the stability of the identity map as an optimal transport from a domain to itself. First, we consider density and domain perturbations near regular portions of domains. More specifically, we find an example of an arbitrarily small Lipschitz (non-convex) perturbation of (a side of) a square that, when taken as the target domain in the optimal transport problem from that square, yields a discontinuous optimal transport. Second, noticing that the discontinuity of optimal transports is an open condition, we find that given any $\varepsilon > 0$, there exists an $\alpha > 0$ and an ε-small $C^{1,\alpha}$ perturbation of a square that produces a discontinuous optimal transport. We then show this is sharp, via an ε-regularity theorem at the boundary (in $n \geq 2$ dimensions), in the sense that given any $\alpha > 0$, there exists an $\varepsilon > 0$ such that any ε-small $C^{1,\alpha}$ perturbation of a $C^{1,\alpha}$ domain has a continuous optimal transport. Second, we investigate density and domains deformations around non-regular parts of domains. Here, we restrict our attention to two dimensions and focus near 90 degree corners. We observe that the ε-regularity theorem we proved on domains "comparable" to half balls can be extended to domains "comparable" to quarter discs. Finally, we show that given two smooth densities f and g on the unit square, the optimal transport taking f to g is of class $C^{2,\alpha}$ up to the boundary for every $\alpha < 1$, yet it may not be of class C^3, even with densities that are arbitrarily C^∞-close to 1.

2. Perturbations in Regular Domains

In [2], Caffarelli showed that the optimal transport ∇u_ε taking the ball $B_1 \subset \mathbb{R}^2$ to the dumbbell $D_\varepsilon := (B^+_{r_\varepsilon} + \mathrm{e}_1) \cup (B^-_{r_\varepsilon} - \mathrm{e}_1) \cup ([-1,1] \times (-\varepsilon, \varepsilon))$, where $r_\varepsilon > 0$ is taken so that $|D_\varepsilon| = |B_1|$, is discontinuous for all $\varepsilon > 0$ sufficiently small.[1] Here, $B^+_r := B_r \cap \{x_1 > 0\}$ and $B^-_r := B_r \cap \{x_1 < 0\}$ for $r > 0$. His example demonstrates the importance of having a convex target in guaranteeing an optimal transport's regularity. A natural follow-up question is, how important is the convexity of the target space in guaranteeing the regularity of the optimal transport? Caffarelli's example and, as far as we know, all known examples suggest that the target domain must be rather far from convex in order to destroy the continuity of the corresponding optimal transport (when, of course, considering the uniform densities on source and target domains). Surprisingly, we shall see that the opposite is true: even a small deviation from convexity can break the continuity of an optimal transport.

Before presenting our examples, let us review Caffarelli's example. Notice that the optimal transport taking B_1 to $D_0 := (B^+_1 + \mathrm{e}_1) \cup (B^-_1 - \mathrm{e}_1)$ is given by

$$\nabla u_0(x) = \begin{cases} x + \mathrm{e}_1 & \text{if } x_1 > 0 \\ x - \mathrm{e}_1 & \text{if } x_1 < 0. \end{cases}$$

By the stability of optimal transports, up to the addition of constants, the potentials u_ε converge locally uniformly (in $\mathbb{R}^2$) to u_0. So since $|\partial u_0(\{(0, \pm 1)\})| = 0$, it follows that

$$\lim_{\varepsilon \to 0} |\partial u_\varepsilon(\{0\} \times (-1,1))| = |\partial u_0(\{0\} \times (-1,1))| = 4.$$

[1] In actuality, he showed that the optimal transport from B_1 to a smoothing of D_ε is discontinuous. However, the regularity of D_ε is irrelevant to the essence of the singular nature of his example.

In turn, we see that the Monge-Ampère measure associated to u_ε must have a singular part for all $\varepsilon > 0$ sufficiently small, that is, ∇u_ε is discontinuous for all $\varepsilon > 0$ sufficiently small. (For a more a hands on explication of Caffarelli's example, one that appeals to the monotonicity of optimal transports, we refer the reader to [6].)

2.1. Lipschitz Perturbations.

Here, we present an example of an ε-Lipschitz perturbation of a square that after smoothing proves the following:

Theorem 2.1. *Given any $\varepsilon > 0$, there exists a smooth, convex domain X and a domain Y that is an ε-small Lipschitz perturbation of X such that the optimal transport taking X to Y is discontinuous.*

Proof. Let

$$X := (0, 4) \times (-2, 2) \quad \text{and} \quad Y := \{(0, 4 + \varepsilon/4) \times (-2, 2)\} \setminus \overline{\Gamma}_\varepsilon$$

where Γ_ε is the interior of the triangle with vertices $(\varepsilon, 0)$, $(0, 1)$, and $(0, -1)$. Notice that X and Y have the same volume and Y is an ε-Lipschitz (non-convex) perturbation of X. If $T = \nabla u$ is the optimal transport taking Y to X, then from [2, 9], we have that u is a strictly convex Alexandrov solution of

$$\det(D^2 u) = 1 \text{ in } Y$$

and of class $C^\infty_{\text{loc}}(Y) \cap C^1(\mathbb{R}^2)$.

Let X' and Y' be the reflections of X and Y over the lines $\{x_2 = 2\}$ and $\{y_2 = 2\}$ respectively and T' be the optimal transport taking Y' to X'. Then, the map $S(y) := RT'(Ry)$ where R is the reflection over the line $\{y_2 = 2\}$ (and also over the line $\{x_2 = 2\}$) is a competing transport map with equal cost. So $S = T'$. Moreover, $T'|_{\overline{Y}} = T$. It follows that $T(\{y_2 = 2\}) \subset \{x_2 = 2\}$. Similarly, considering reflections of X and Y over the lines $\{x_1 = 4\}$ and $\{y_1 = 4 + \varepsilon/4\}$ and the lines $\{x_2 = -2\}$ and $\{y_2 = -2\}$, reflections of $X^+ := X \cap \{x_2 > 0\}$ and $Y^+ := Y \cap \{y_2 > 0\}$ over the lines $\{x_1 = 0\}$ and $\{y_1 = 0\}$, and reflections of $X^- := X \cap \{x_2 < 0\}$ and $Y^- := Y \cap \{y_2 < 0\}$ over the lines $\{x_1 = 0\}$ and $\{y_1 = 0\}$, we deduce that $T = \nabla u$ maps $\{4+\varepsilon/4\} \times [-2, 2]$ homeomorphically to $\{4\} \times [-2, 2]$, maps $[0, 4 + \varepsilon/4] \times \{\pm 2\}$ homeomorphically to $[0, 4] \times \{\pm 2\}$, and maps $\{0\} \times [1, 2]$ and $\{0\} \times [-2, -1]$ homeomorphically to subsegments of $\{0\} \times [0, 2]$ and $\{0\} \times [-2, 0]$ respectively. Also, by symmetry and restriction, u is strictly convex on $\overline{Y} \cap \{y_2 \geq 0\}$ and $\overline{Y} \cap \{y_2 \leq 0\}$ (see [3]).

There are two possibilities. Either $\nabla u(\partial Y) = \partial X$ and u is strictly convex on $\overline{Y}$ or some portion (symmetric with respect to the y_1-axis) of the left boundary of Y will map inside X. In particular, in the second scenario, a symmetric subset of the two segments joining $(0, 1)$, $(\varepsilon, 0)$, and $(0, -1)$ and containing the point $(\varepsilon, 0)$ will map to a segment along $X \cap \{x_2 = 0\}$, and the optimal transport ∇u^*, where u^* is the Legendre transform of u, taking X to Y will be discontinuous along the segment joining $(0, 0)$ and $(0, t_\varepsilon)$ where $\nabla u(\varepsilon, 0) = (0, t_\varepsilon)$. (See Figure 2.1.)

Suppose that $\nabla u(\partial Y) = \partial X$. Then, taking the partial Legendre transform of u in the e_1-direction and setting $v = \partial_1 u^\star$, we find that

$$\begin{cases} \Delta v = 0 & \text{in } X \\ v = h(p) & \text{on } (0, 4) \times \{\pm 2\} \\ v = 4 + \varepsilon/4 & \text{on } \{4\} \times (-2, 2) \\ v = \max\{0, -\varepsilon|x_2| + \varepsilon\} & \text{on } \{0\} \times (-2, 2) \end{cases}$$

where h is an increasing function such that $h(0) = 0$ and $h(4) = 4 + \varepsilon/4$. (See the proof of Theorem 3.3 for details on the partial Legendre transform.) Consider the harmonic function

$$b(p, x_2) := \varepsilon\frac{2}{\pi}\Re(z \log z) + \varepsilon + 2(x_2^2 - p^2) + 16p$$

where $z = p + ix_2$ and $\Re(z \log z)$ denotes the real part of $z \log z$. Observe that b is an upper barrier for v. Thus, as $b(0) = v(0)$ and $b(p, 0) - b(0, 0) < 0$ for all $p > 0$ sufficiently small, we deduce that

$$\int_0^p \partial_1 v(t, 0)\,\mathrm{d}t < 0$$

for all $0 < p \ll 1$. It follows that

$$\partial_{11}u^\star(p, 0) = \partial_1 v(p, 0) < 0$$

for all sufficiently small $p > 0$. But this contradicts the convexity of $u^\star$ in the e_1-direction. □

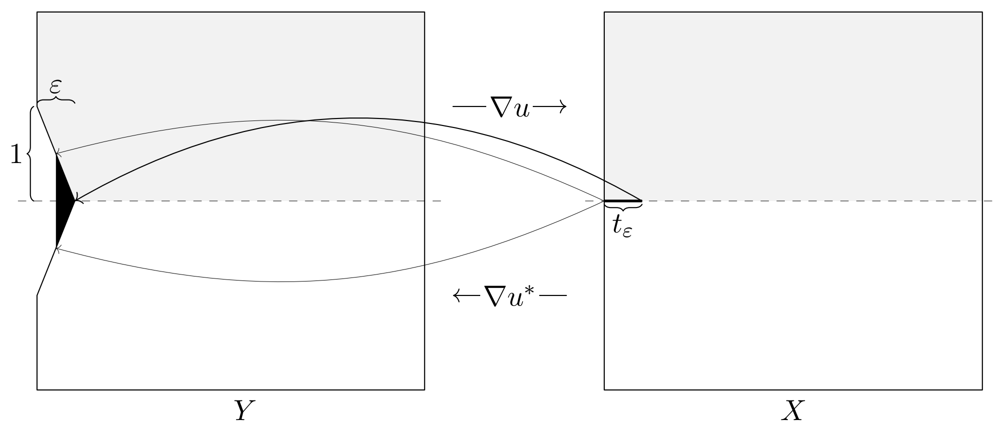

FIGURE 2.1. The optimal transport ∇u^* splits mass.

2.2. $C^{1,\alpha}$ **Perturbations.** Notice that the proof of Caffarelli's optimal transport's discontinuity shows that the discontinuity of optimal transports is an open condition. More precisely, by the stability of optimal transports, we have the following lemma:

Lemma 2.2. *Let $X_0, Y_0 \subset \mathbb{R}^n$ be open, bounded sets and $X_\varepsilon, Y_\varepsilon \subset \mathbb{R}^n$ be a sequences of open, bounded sets such that $\operatorname{dist}(\partial X_\varepsilon, \partial X_0) + \operatorname{dist}(\partial Y_\varepsilon, \partial Y_0) \to 0$ as $\varepsilon \to 0$. Let f_ε and g_ε be sequences of densities uniformly bounded away from zero and infinity on X_ε and Y_ε respectively, satisfying the mass balance condition $\|f_\varepsilon\|_{L^1(X_\varepsilon)} = \|g_\varepsilon\|_{L^1(Y_\varepsilon)}$, and such that $f_\varepsilon \to f_0$ and $g_\varepsilon \to g_0$ in L^1 as $\varepsilon \to 0$. If the optimal transport ∇u_0 taking f_0 to g_0 is discontinuous, then there exists an $\varepsilon_0 > 0$ such that the optimal transport ∇u_ε taking f_ε to g_ε is discontinuous for all $\varepsilon < \varepsilon_0$.*

Lemma 2.2 allows us to extend Theorem 2.1 to small $C^{1,\alpha}$ perturbations. In particular, replace the left boundary of the target domain in Theorem 2.1, which is given by the graph of the function $\max\{0, -\varepsilon|y_2| + \varepsilon\}$, with the graph of the function $\max\{0, -\varepsilon|y_2|^{1+\alpha} + \varepsilon\}$. Then, smooth out the graph of $\max\{0, -\varepsilon|y_2|^{1+\alpha} + \varepsilon\}$ near $y_2 = \pm 1$. Now adjust the width of the lower and upper boundaries of our deformed square so that the domain has total mass 16. Lemma 2.2 implies that if $\alpha = \alpha(\varepsilon) > 0$ is sufficiently small, then the optimal transport from our original square to this new non-convex domain is discontinuous. So after smoothing, we have the following:

Theorem 2.3. *Given any $\varepsilon > 0$, there exists an $\alpha = \alpha(\varepsilon) > 0$, a smooth, convex domain X, and a domain Y that is an ε-small $C^{1,\alpha}$ perturbation of X such that the optimal transport taking X to Y is discontinuous.*

That said, $C^{1,\alpha}$ is the borderline topology in which small perturbations can break an optimal transport's continuity. More precisely, Theorem 2.3 is sharp in view of Theorem 2.4.

Theorem 2.4. *Let X be a $C^{1,\alpha}$ domain, Y be an ε-small $C^{1,\alpha}$ perturbation of X, and ∇u be the optimal transport taking X to Y. If $\varepsilon = \varepsilon(\alpha) > 0$ is sufficiently small, then $\nabla u : \overline{X} \to \overline{Y}$ is a bi-Hölder continuous homeomorphism.*

Let $x \in \mathbb{R}^n$ be given by $x = (x_0, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathcal{B}_R := \{x_0 \in \mathbb{R}^{n-1} : |x_0| < R\} = B_R \cap \{x_n = 0\}$.

Proof. Up to a translation and a rotation, we assume that $0 \in \partial X$ and $\{x_n = 0\}$ is a supporting plane to X at 0. That is,

$$X \cap B_\rho \subset \{x_n \geq \gamma(x_0)\}, \quad \partial X \cap B_\rho = \{x_n = \gamma(x_0)\}, \quad \text{and} \quad \gamma(0) = \nabla\gamma(0) = 0$$

where $\gamma \in C^{1,\alpha}(\mathcal{B}_\rho)$ defines ∂X in B_ρ. Up to a translation, we have that $0 \in \partial Y$. After a shearing (affine) transformation, we may assume that $\{y_n = 0\}$ is tangent to Y at 0. In particular,

$$Y \cap B_\rho \subset \{y_n \geq \zeta(y_0)\}, \quad \partial Y \cap B_\rho = \{y_n \geq \zeta(y_0)\}, \quad \text{and} \quad \zeta(0) = \nabla\zeta(0) = 0$$

where $\zeta \in C^{1,\alpha}(\mathcal{B}_\rho)$ defines ∂Y in B_ρ. By the stability of optimal transports, up to the addition of a constant, we have that

$$\left\| u(x) - \frac{1}{2}|x|^2 \right\|_{L^\infty(B_{\rho/8} \cap \{x_n \geq \gamma(x_0)\})} \leq \tau$$

where $\tau = \tau(\varepsilon) \to 0$ as $\varepsilon \to 0$. So after quadratically rescaling, i.e., considering $u_h(x) := u(hx)/h^2$, $X_h := X/h$, and $Y_h := Y/h$, we find that $(\nabla u_h)_\# \mathbf{1}_{X_h} = \mathbf{1}_{Y_h}$ and

$$\left\| u_h(x) - \frac{1}{2}|x|^2 \right\|_{L^\infty(B_{\rho/8h} \cap \{x_n \geq \gamma_h(x_0)\})} \leq \frac{\tau}{h^2}.$$

Also,

$$[\nabla\gamma_h]_{C^{0,\alpha}(\mathcal{B}_{\rho/h})} + [\nabla\zeta_h]_{C^{0,\alpha}(\mathcal{B}_{\rho/h})} \leq Ch^\alpha,$$

where γ_h and ζ_h define the boundaries of X_h and Y_h in $B_{\rho/h}$ respectively and $C > 0$ is such that $[\nabla\gamma]_{C^{0,\alpha}(\mathcal{B}_\rho)} + [\nabla\zeta]_{C^{0,\alpha}(\mathcal{B}_\rho)} \leq C$. In particular, $\gamma_h(x) = \gamma(hx)/h$ and $\zeta_h(x) = \zeta(hx)/h$. Now take $h > 0$ so that $Ch^\alpha \leq \delta$. Next, choose $\rho > 0$, smaller if needed, so that $\rho = 4h$. Lastly, take $\varepsilon > 0$ small enough so that $\tau/h^2 \leq \eta$. Then, restricting our attention to $\mathcal{C} = \overline{B}_1 \cap \overline{X}_h$ and $\mathcal{K} = \partial u_h(\overline{B}_1) \cap \overline{Y}_h$, we can apply Theorem 2.5 with $f = \mathbf{1}_{\mathcal{C}}$ and $g = \mathbf{1}_{\mathcal{K}}$ to conclude that $u_h \in C^{1,\beta}(B_r \cap \{x_n \geq \gamma_h(x_0)\})$. As 0 represents an arbitrary point on the boundary of X, scaling back together with a simple covering argument concludes the proof of the theorem. □

The following theorem, the main ingredient in the proof of Theorem 2.4 above, is a generalization of the ε-regularity theorem at the boundary for C^2 domains proved by Chen and Figalli ([5, Theorem 2.1]) to $C^{1,\alpha}$ domains. Heuristically, since $C^{1,\alpha}$ domains, like

C^2 domains, flatten under dilations, we might expect that extending their arguments to our setting is rather simple. However, in practice, our situation is quite delicate, and some additional details and new ideas must be presented and developed. That said, for an explanation of any estimate or computation that does not specifically see the difference between a C^2 and $C^{1,\alpha}$ boundary, we refer the reader to their proof or the proofs of [7, Theorem 4.3] and [10, Proposition 4.2].

Theorem 2.5. *Let $\mathcal{C}$ and $\mathcal{K}$ be two closed subsets of $\mathbb{R}^n$ such that*

$$B_{1/2} \cap \{x_n \geq \gamma(x_0)\} \subset \mathcal{C} \subset B_2 \cap \{x_n \geq \gamma(x_0)\}$$

and

$$B_{1/2} \cap \{y_n \geq \zeta(y_0)\} \subset \mathcal{K} \subset B_2 \cap \{y_n \geq \zeta(y_0)\}$$

where

$$\gamma, \zeta \in C^{1,\alpha}(\mathcal{B}_4), \quad \gamma(0) = \zeta(0) = 0, \quad \textit{and} \quad \nabla\gamma(0) = \nabla\zeta(0) = 0. \tag{2.1}$$

Let u be a convex potential such that $(\nabla u)_\# f = g$ for two densities f and g supported on $\mathcal{C}$ and $\mathcal{K}$ respectively. Given $\beta \in (0,1)$, there exist constants $r, \eta, \delta > 0$, with $\delta = \delta(\eta)$ and $\eta = \eta(\alpha, \beta, n)$, such that the following holds: if

$$[\nabla\gamma]_{C^{0,\alpha}(\mathcal{B}_4)} + [\nabla\zeta]_{C^{0,\alpha}(\mathcal{B}_4)} \leq \delta, \tag{2.2}$$

$$\|f - \mathbf{1}_{\mathcal{C}}\|_\infty + \|g - \mathbf{1}_{\mathcal{K}}\|_\infty \leq \delta, \tag{2.3}$$

and

$$\left\| u(x) - \frac{1}{2}|x|^2 \right\|_{L^\infty(B_{1/2} \cap \{x_n \geq \gamma(x_0)\})} \leq \eta, \tag{2.4}$$

then $u \in C^{1,\beta}(B_r \cap \{x_n \geq \gamma(x_0)\})$.

In what follows, we let C and c be generic positive constants that may change from line to line. Their dependencies, if any, will either be clear from context or explicitly given.

Before proceeding with the proof of Theorem 2.5, let us make a remark and an associated definition. From the point of view of optimal transportation, the cost $-x \cdot y$ is the same as $-x \cdot y|_{\operatorname{spt} f \times \operatorname{spt} g}$. So we shall often work with the intersection of the subdifferential of our convex potential with the support of our target measure. In particular, we define

$$\partial_* u(x) := \partial u(x) \cap \operatorname{spt} g \quad \text{and} \quad \partial_* u(E) := \bigcup_{x \in E} \partial_* u(x)$$

when $(\nabla u)_\# f = g$.

Proof. For clarity's sake, we divide the proof into several steps.

– *Step 1: An initial normalization.*

Let

$$x^0 \in B_r \cap \{x_n \geq \gamma(x_0)\} \quad \text{and} \quad y^0 \in \partial_* u(x^0).$$

Thus, using (2.1) and (2.2), we have that

$$y_n^0 \geq \zeta(y_0^0) \geq -4\delta. \tag{2.5}$$

Notice that

$$w(x) := u(x) - \frac{1}{2}|x|^2 + \frac{1}{2}|x - x^0|^2$$

is convex and $y^0 - x^0 \in \partial w(x^0)$. Hence, by (2.4), (2.1), and (2.2), we deduce that for $\mathrm{e} \in \mathbb{S}^{n-1}$ such that $\angle(\mathrm{e}, \mathrm{e}_n) = \pi/4$,

$$(y^0 - x^0) \cdot \mathrm{e} \leq \frac{w(x^0 + \eta^{1/2}\mathrm{e}) - w(x^0)}{\eta^{1/2}} \leq \frac{5}{2}\eta^{1/2} \tag{2.6}$$

provided that $r + \eta^{1/2} < 1/2$. The same estimate holds with $\mathrm{e} = \mathrm{e}_n$. If $x_n^0 - \gamma(x_0^0) \geq \eta^{1/2}$, then $x^0 - \eta^{1/2}\mathrm{e}_n \in B_{1/2} \cap \{x_n \geq \gamma(x_0)\}$. So using w as before, we find that

$$(y^0 - x^0) \cdot (-\mathrm{e}_n) \leq \frac{5}{2}\eta^{1/2}.$$

If, on the other hand, $x_n^0 - \gamma(x_0^0) < \eta^{1/2}$, then

$$(y^0 - x^0) \cdot (-\mathrm{e}_n) \leq \eta^{1/2} + (4 + r)\delta \leq C(\eta^{1/2} + \delta),$$

recalling (2.5) and noticing that $|\gamma(x_0^0)| \leq \delta r$ by (2.1) and (2.2). As every $\theta \in \mathbb{S}^{n-1}$ can be written as a linear combination of $\pm\mathrm{e}_n$ and some e such that $\angle(\mathrm{e}, \mathrm{e}_n) = \pi/4$ with positive coefficients, it follows that

$$|y^0 - x^0| \leq C(\eta^{1/2} + \delta). \tag{2.7}$$

First, consider the change of variables

$$\hat{x} := x - x^0 \quad \text{and} \quad \hat{y} := y - y^0.$$

Notice that

$$\hat{\gamma}(\hat{x}_0) := \gamma(\hat{x}_0 + x_0^0) - x_n^0 \quad \text{and} \quad \hat{\zeta}(\hat{y}_0) := \zeta(\hat{y}_0 + y_0^0) - y_n^0$$

define the lower boundaries of $\hat{\mathcal{C}} := \mathcal{C} - x^0$ and $\hat{\mathcal{K}} := \mathcal{K} - y^0$ respectively and, by (2.1), (2.2), and (2.7),

$$0 \geq \hat{\zeta}(0) > -C(\eta^{1/2} + \delta).$$

Furthermore, using (2.4) and (2.7), we have that

$$\left\| \hat{u}(\hat{x}) - \frac{1}{2}|\hat{x}|^2 \right\|_{L^\infty(B_{5/12} \cap \{\hat{x}_n \geq \hat{\gamma}(\hat{x}_0)\})} \leq C(\eta^{1/2} + \delta)$$

where

$$\hat{u}(\hat{x}) := u(x) - u(x^0) - y^0 \cdot (x - x^0).$$

Also, defining $\hat{f}(\hat{x}) := f(\hat{x} + x^0)$ and $\hat{g}(\hat{y}) := g(\hat{y} + y^0)$, it is clear that $(\nabla\hat{u})_\# \hat{f} = \hat{g}$.

– *Case 1:* $x^0 \in \{x_n = \gamma(x_0)\}$.

Let R be the rotation matrix that makes the tangent line to $\hat{\mathcal{C}}$ at 0 (which is on the lower boundary of $\hat{\mathcal{C}}$) horizontal, and consider the change of coordinates

$$\bar{x} := R\hat{x} \quad \text{and} \quad \bar{y} := (R^*)^{-1}\hat{y}.$$

By (2.1) and (2.2), we see that $|\nabla\hat{\gamma}(0)| \leq \delta r^\alpha$. Therefore, the angle defining R is smaller than δr^α. So letting $\bar{\gamma}$ and $\bar{\zeta}$ define the lower boundaries of $\bar{\mathcal{C}} := R\hat{\mathcal{C}}$ and $\bar{\mathcal{K}} := (R^*)^{-1}\hat{\mathcal{K}}$ respectively, it follows that

$$[\nabla\bar{\gamma}]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\bar{\zeta}]_{C^{0,\alpha}(\mathcal{B}_3)} \leq C\delta$$

and

$$0 \geq \bar{\zeta}(0) > -C(\eta^{1/2} + \delta).$$

Furthermore, letting

$$\bar{u}(\bar{x}) := \hat{u}(R^{-1}\bar{x}), \quad \bar{f}(\bar{x}) := \hat{f}(R^{-1}\bar{x}), \quad \text{and} \quad \bar{g}(\bar{y}) := \hat{g}(R^{*}\bar{y}),$$

we have that

$$\left\| \bar{u}(\bar{x}) - \frac{1}{2}|\bar{x}|^2 \right\|_{L^\infty(B_{5/12} \cap \{\bar{x}_n \geq \bar{\gamma}(\bar{x}_0)\})} \leq C(\eta^{1/2} + \delta)$$

and $(\nabla \bar{u})_\# \bar{f} = \bar{g}$.

Finally, define the change of variables

$$\check{x} := N\bar{x} \quad \text{and} \quad \check{y} := (N^*)^{-1}\bar{y}$$

with

$$N\bar{z} := \bar{z} + (\nabla\bar{\zeta}(0), 0)\bar{z}_n.$$

If we let $\check{\gamma}$ and $\check{\zeta}$ define the lower boundaries of $\check{\mathcal{C}} := N\bar{\mathcal{C}}$ and $\check{\mathcal{K}} := (N^*)^{-1}\bar{\mathcal{K}}$ respectively, then

$$\nabla\check{\gamma}(0) = \nabla\check{\zeta}(0) = 0.$$

Moreover,

$$[\nabla\check{\gamma}]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\check{\zeta}]_{C^{0,\alpha}(\mathcal{B}_3)} \leq C\delta.$$

Also, note that

$$|N - \mathrm{Id}\,| \leq C\delta$$

provided η, δ, and r are sufficiently small.

– *Case 2:* $x^0 \in \{x_n > \gamma(x_0)\}$.

From (2.1) and (2.2), we see that the angle between the $\hat{x}_n$-axis and the line through the origin that meets $\{\hat{x}_n = \hat{\gamma}(\hat{x}_0)\}$ orthogonally is at most 4δ. So let R be the rotation matrix that makes this line vertical and consider the change of coordinates

$$\check{x} := R\hat{x} \quad \text{and} \quad \check{y} := (R^*)^{-1}\hat{y}.$$

Then,

$$[\nabla\check{\gamma}]_{C^{0,\alpha}(\mathcal{B}_3])} + [\nabla\check{\zeta}]_{C^{0,\alpha}(\mathcal{B}_3)} \leq C\delta$$

where $\check{\gamma}$ and $\check{\zeta}$ define the lower boundaries of $\check{\mathcal{C}} := R\hat{\mathcal{C}}$ and $\check{\mathcal{K}} := (R^*)^{-1}\hat{\mathcal{K}}$ respectively.

In summary, if we define the potential

$$\check{u}(\check{x}) := \bar{u}(N^{-1}\check{x}) \quad \text{or} \quad \check{u}(\check{x}) := \hat{u}(R^{-1}\check{x})$$

and the densities

$$\check{f}(\check{x}) := \bar{f}(N^{-1}\check{x}) \quad \text{and} \quad \check{g}(\check{y}) := \bar{g}(N^*\check{y}) \quad \text{or} \quad \check{f}(\check{x}) := \hat{f}(R^{-1}\check{x}) \quad \text{and} \quad \check{g}(\check{y}) := \hat{g}(R^*\check{y})$$

depending on whether we are in Case 1 or Case 2, then

$$(\nabla\check{u})_\# \check{f} = \check{g},$$

and provided that r, δ, and η are sufficiently small,

$$B_{1/3} \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\} \subset \check{\mathcal{C}} \subset B_3 \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\}$$

and

$$B_{1/3} \cap \{\check{y}_n \geq \check{\zeta}(\hat{y}_0)\} \subset \check{\mathcal{K}} \subset B_3 \cap \{\check{y}_n \geq \check{\zeta}(\check{y}_0)\}$$

with

$$[\nabla\check{\gamma}]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\check{\zeta}]_{C^{0,\alpha}(\mathcal{B}_3)} \leq \check{\delta}.$$

If $x_n^0 = \gamma(x_0^0)$, then

$$\check{\gamma}(0) = 0, \quad -(\check{\eta} + \check{\delta}) \leq \check{\zeta}(0) \leq 0, \quad \text{and} \quad \nabla\check{\gamma}(0) = \nabla\check{\zeta}(0) = 0;$$

where as if $x_n^0 > \gamma(x_0^0)$, then

$$-r < \check{\gamma}(0) \leq 0, \quad -(\check{\eta} + \check{\delta} + r) < \check{\zeta}(0) \leq 0, \quad \nabla\check{\gamma}(0) = 0, \quad \text{and} \quad |\nabla\check{\zeta}(0)| \leq \check{\delta}.$$

Furthermore,

$$\left\| \check{u}(\check{x}) - \frac{1}{2}|\check{x}|^2 \right\|_{L^\infty(B_{1/3} \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\})} \leq \check{\eta}, \tag{2.8}$$

and by (2.3),

$$\|\check{f} - \mathbf{1}_{\check{\mathcal{C}}}\|_\infty + \|\check{g} - \mathbf{1}_{\check{\mathcal{K}}}\|_\infty \leq \check{\delta}.$$

Here, $\check{\delta} \to 0$ as $\delta \to 0$ and $\check{\eta} \to 0$ as $\eta^{1/2} + \delta \to 0$.

– *Step 2: Finding and estimating a smooth approximation of $\check{u}$.*

We begin with an important lemma.

Lemma 2.6. *Let $\mathcal{C}$ and $\mathcal{K}$ be two closed subsets of $\mathbb{R}^n$ such that*

$$B_{1/R} \cap \{x_n \geq \gamma(x_0)\} \subset \mathcal{C} \subset B_R \cap \{x_n \geq \gamma(x_0)\}$$

and

$$B_{1/R} \cap \{y_n \geq \zeta(y_0)\} \subset \mathcal{K} \subset B_R \cap \{y_n \geq \zeta(y_0)\}$$

where γ and ζ are of class $C^{1,\alpha}(\mathcal{B}_R)$ and such that

$$-\frac{1}{R^3} \leq \gamma(0), \zeta(0) \leq 0 \quad \text{and} \quad |\nabla\gamma(0)|, |\nabla\zeta(0)| \leq \delta.$$

Let

$$l_\gamma := |\gamma(0)| + |\nabla\gamma(0)|R + [\nabla\gamma]_{C^{0,\alpha}(\mathcal{B}_R)}R^2$$

and

$$l_\zeta := |\zeta(0)| + |\nabla\zeta(0)|R + [\nabla\zeta]_{C^{0,\alpha}(\mathcal{B}_R)}R^2,$$

and define

$$\mathcal{C}_+ := \mathcal{C} \cup (B_{1/R} \cap \{x_n \geq -l_\gamma\}) \quad \text{and} \quad \mathcal{K}_+ := \mathcal{K} \cup (B_{1/R} \cap \{y_n \geq -l_\zeta\}).$$

Suppose u is a convex function such that $(\nabla u)_\# f = g$ for two densities f and g supported on $\mathcal{C}$ and $\mathcal{K}$ respectively. Let $\lambda > 0$ be such that $|\mathcal{C}_+| = |\lambda\mathcal{K}_+|$, where $\lambda\mathcal{K}_+$ denotes the dilation of $\mathcal{K}_+$ with respect to the origin, and let v be a convex function such that $v(0) = u(0)$ and $(\nabla v)_\# \mathbf{1}_{\mathcal{C}_+} = \mathbf{1}_{\lambda\mathcal{K}_+}$. In addition, let u^* and v^* be such that $u^*(0) = v^*(0)$, $(\nabla u^*)_\# g = f$, and $(\nabla v^*)_\# \mathbf{1}_{\lambda\mathcal{K}_+} = \mathbf{1}_{\mathcal{C}_+}$. Then, there exists a nonnegative, increasing function $\omega = \omega(\delta)$, depending on R, such that $\omega(\delta) \geq \delta$, $\omega(0^+) = 0$, and the following holds: if

$$[\nabla\gamma]_{C^{0,\alpha}(\mathcal{B}_R)} + [\nabla\zeta]_{C^{0,\alpha}(\mathcal{B}_R)} \leq \delta$$

and

$$\|f - \mathbf{1}_{\mathcal{C}}\| + \|g - \mathbf{1}_{\mathcal{K}}\| \leq \delta,$$

then

$$\|u - v\|_{L^\infty(B_{1/R} \cap \mathcal{C})} + \|u^* - v^*\|_{L^\infty(B_{1/R^2} \cap \mathcal{K})} \leq \omega(\delta).$$

Proof of Lemma 2.6. The proof is identical to that of [5, Lemma 4.1]. □

Choose $\lambda > 0$ such that $|\check{\mathcal{C}}_+| = |\lambda\check{\mathcal{K}}_+|$, where $\check{\mathcal{C}}_+$ and $\check{\mathcal{K}}_+$ are defined as in Lemma 2.6, and let $\check{v}$ be a convex function such that $(\nabla\check{v})_\#\mathbf{1}_{\check{\mathcal{C}}_+} = \mathbf{1}_{\lambda\check{\mathcal{K}}_+}$ and $\check{v}(0) = \check{u}(0) = 0$. By Lemma 2.6,

$$\|\check{u} - \check{v}\|_{L^\infty(B_{1/3}\cap\{\check{x}_n \geq \check{\gamma}(\check{x}_0)\})} \leq \omega(\check{\delta}). \tag{2.9}$$

Define $\check{\mathcal{C}}_+'$ to be the reflection of $\check{\mathcal{C}}_+$ over the hyperplane $\{\check{x}_n = -l_{\check{\gamma}}\}$ and $(\lambda\check{\mathcal{K}}_+)'$ to be the reflection of $\lambda\check{\mathcal{K}}_+$ over the hyperplane $\{\check{y}_n = -\lambda l_{\check{\zeta}}\}$. If $\check{v}'$ is a convex potential whose gradient is the optimal transport taking $\mathbf{1}_{\check{\mathcal{C}}_+'}$ to $\mathbf{1}_{(\lambda\check{\mathcal{K}}_+)'}$, then, by symmetry, $\nabla\check{v}'|_{\check{\mathcal{C}}_+} = \nabla\check{v}$. Also,

$$\nabla\check{v}'(\{\check{x}_n = -l_{\check{\gamma}}\}) \subset \{\check{y}_n = -\lambda l_{\check{\zeta}}\}. \tag{2.10}$$

Without loss of generality, $\check{v}'(0) = \check{v}(0) = 0$. Therefore, by (2.8) and (2.9), it follows that

$$\left\|\check{v}'(\check{x}) - \frac{1}{2}|\check{x}|^2\right\|_{L^\infty(B_{1/3}\cap\{\check{x}_n \geq \check{\gamma}(\check{x}_0)\})} \leq \check{\eta} + \omega(\check{\delta}).$$

Then, by symmetry and the convexity of $\check{v}'$, we deduce that

$$\left\|\check{v}'(\check{x}) - \frac{1}{2}|\check{x}|^2\right\|_{L^\infty(B_{1/6})} \leq C_0(\check{\eta} + \omega(\check{\delta}) + |\check{\gamma}(0)| + |\nabla\check{\gamma}(0)| + |\check{\zeta}(0)| + |\nabla\check{\zeta}(0)|). \tag{2.11}$$

So provided $\check{\eta}$, $\check{\delta}$, and r are sufficiently small (recall the lines just before (2.8)), arguing as in [7, Theorem 4.3, Step 1], we have that

$$\det(D^2\check{v}') = 1 \text{ in } B_{1/7}, \tag{2.12}$$

in the Alexandrov sense, and $\check{v}'$ is uniformly convex and smooth inside $B_{1/8}$. In particular,

$$\|\check{v}'\|_{C^3(B_{1/8})} \leq C_1 \quad \text{and} \quad \frac{1}{C_1}\,\mathrm{Id} \leq D^2\check{v}' \leq C_1\,\mathrm{Id} \text{ in } B_{1/8}. \tag{2.13}$$

Moreover,

$$Z_{\check{v}'}(h) := \{\check{x} : \check{v}'(\check{x}) \leq \nabla\check{v}'(0)\cdot\check{x} + h\} \Subset B_{1/9} \quad \forall h \leq \check{h} \tag{2.14}$$

where $\check{h} > 0$ is a small constant.

From now on, we will not distinguish $\check{v}'$ and $\check{v}$.

Let us estimate $\nabla\check{v}(0)$ and $D^2\check{v}(0)$. Arguing as we did to prove (2.7), considering the convex function

$$\check{w}(\check{x}) := \check{u}(\check{x}) - \check{v}(\check{x}) + \frac{C_1}{2}|\check{x}|^2,$$

we deduce that

$$|\nabla\check{v}(0)| \leq C_2\omega(\check{\delta})^{1/2}. \tag{2.15}$$

By (2.10), $\partial_n\check{v}$ is constant on $\{\check{x}_n = -l_{\check{\gamma}}\}$, from which we infer that

$$\partial_{in}\check{v}|_{\{\check{x}_n = -l_{\check{\gamma}}\}} \equiv 0 \quad \forall i = 1,\dots,n-1. \tag{2.16}$$

Also, using (2.13), we find that

$$|\partial_{ij}\check{v}(0) - \partial_{ij}\check{v}(0, -l_{\check{\gamma}})| \leq C_1 l_{\check{\gamma}}. \tag{2.17}$$

– *Step 3: Estimating the eccentricity of* $\check{u}$.

Let

$$A := [D^2\check{v}(0)]^{-1/2}.$$

Taylor expanding $\check{v}$ around the origin, recalling that $\check{v}(0) = \check{u}(0) = 0$, and using (2.9), (2.13), and (2.15), we see that

$$\begin{aligned} &\left\| \check{u}(\check{x}) - \frac{1}{2}|A^{-1}\check{x}|^2 \right\|_{L^\infty(AB_{(16h)^{1/2}} \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\})} \\ &\leq \omega(\check{\delta}) + 4C_2(C_1\omega(\check{\delta})h)^{1/2} + 64C_1^{5/2}h^{3/2} \leq \frac{1}{2}\check{\eta}h \end{aligned} \tag{2.18}$$

first choosing h and then choosing $\check{\delta}$ also sufficiently small depending on $\check{\eta}$. Similarly, first choosing h and then choosing $\check{\delta}$ also sufficiently small depending on $\check{\eta}$, we find that

$$\left\| \check{u}^*(\check{y}) - \frac{1}{2}|A\check{y}|^2 \right\|_{L^\infty(A^{-1}B_{(16h)^{1/2}} \cap \{\check{y}_n \geq \check{\zeta}(\check{y}_0)\})} \leq \frac{1}{2}\check{\eta}h.$$

Now notice that the sections of $\check{u}$ and $\check{v}$ appropriately restricted are comparable if $\check{\delta}$ is sufficiently small. In particular, choose $h \leq 2\check{h}/3$ and take $\check{\delta}$ small enough so that $(C_2+1)\omega(\check{\delta})^{1/2} \leq h/2$. Then, recalling (2.14), we have that

$$Z_{\check{v}}(h/2) \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\} \subset S_{\check{u}}(h) \subset Z_{\check{v}}(3h/2) \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\} \Subset B_{1/9} \tag{2.19}$$

where

$$S_{\check{u}}(h) := \{\check{x} \in \check{\mathcal{C}} : \check{u}(\check{x}) \leq h\}.$$

Furthermore, using (2.13), we deduce that

$$\mathcal{E}(h) \subset Z_{\check{v}}(h + C_1(2C_1h)^{3/2}) \quad \text{and} \quad Z_{\check{v}}(h) \subset \mathcal{E}(h + C_1(2C_1h)^{3/2})$$

where

$$\mathcal{E}(h) := \left\{ \check{x} : \frac{1}{2}D^2\check{v}(0)\check{x}\cdot\check{x} < h \right\} = AB_{(2h)^{1/2}}.$$

Hence, (2.19) implies that

$$AB_{(h/7)^{1/2}} \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\} \subset S_{\check{u}}(h) \subset AB_{(7h)^{1/2}} \cap \{\check{x}_n \geq \check{\gamma}(\check{x}_0)\} \tag{2.20}$$

if $\check{\delta}$ and h are small enough. In addition, arguing as in the proof of [7, Theorem 4.3], we see that

$$A^{-1}B_{(h/7)^{1/2}} \cap \{\check{y}_n \geq \check{\zeta}(\check{y}_0)\} \subset \partial_*\check{u}(S_{\check{u}}(h)) \subset A^{-1}B_{(7h)^{1/2}} \cap \{\check{y}_n \geq \check{\zeta}(\check{y}_0)\}, \tag{2.21}$$

provided that $\check{\delta}$ and h are sufficiently small.

– *Case 1:* $x^0 \in \{x_n = \gamma(x_0)\}$, *i.e.,* $\check{\gamma}(0) = 0$.

– *Step 4.1: A change of variables.*

Recalling that A is symmetric, $\det(A) = 1$, $C_1^{-1/2}\,\mathrm{Id} \leq A \leq C_1^{1/2}$, (2.16), and (2.17), a simple computation shows that there exists a matrix M such that

$$\det(M) = 1,$$

the matrix $M^{-1}A^{-1}$ is symmetric,

$$(M^{-1}A^{-1})_{ni} = (M^{-1}A^{-1})_{in} = 0 \quad \forall i = 1, \dots, n-1,$$

and

$$|M - \mathrm{Id}\,| \leq C_3\check{\delta}.$$

(In this case, the $\check{\delta}$ factor comes from the Hölder semi-norm of the gradient of $\check{\gamma}$ only.) Now consider the change of variables

$$\tilde{x} = \frac{1}{h^{1/2}} M^{-1} A^{-1} \check{x} \quad \text{and} \quad \tilde{y} := \frac{1}{h^{1/2}} M^* A \check{y}.$$

Let

$$u_1(\tilde{x}) := \frac{1}{h} \check{u}(h^{1/2} A M \tilde{x}),$$

and set

$$\mathcal{C}_1 := S_{u_1}(1) \quad \text{and} \quad \mathcal{K}_1 := \partial_* u_1(S_{u_1}(1)).$$

As $\det(M) = \det(A) = 1$, we deduce that

$$(\nabla u_1)_\# f_1 = g_1$$

with

$$f_1(\tilde{x}) := \check{f}(h^{1/2} A M \tilde{x}) \mathbf{1}_{\mathcal{C}_1} \qquad \text{and} \qquad g_1(\tilde{y}) := \check{g}(h^{1/2} A^{-1} (M^*)^{-1} \tilde{y}) \mathbf{1}_{\mathcal{K}_1}.$$

Then, from (2.20), (2.21), and our estimate on M,

$$B_{1/3} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset \mathcal{C}_1 \subset B_3 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\}$$

and

$$B_{1/3} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\} \subset \mathcal{K}_1 \subset B_3 \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\}.$$

Additionally,

$$\gamma_1(0) = 0, \quad \frac{1}{C_4 h^{1/2}} |\check{\zeta}(0)| \leq |\zeta_1(0)| \leq \frac{C_4}{h^{1/2}} |\check{\zeta}(0)|, \quad \nabla\gamma_1(0) = \nabla\zeta_1(0) = 0,$$

and

$$[\nabla\gamma_1]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\zeta_1]_{C^{0,\alpha}(\mathcal{B}_3)} \leq c_0 \check{\delta} =: \delta_1$$

for $c_0 < 1$, choosing h and $\check{\delta}$ small enough so that $C_4 h^{\alpha/2} \ll 1$.

– *Step 5.1: An iteration scheme.*

In order run Steps 2 through 4.1 on u_1, $\mathcal{C}_1$, $\mathcal{K}_1$, γ_1, ζ_1, f_1, and g_1, we need to ensure two things: 1. that the hypotheses of Lemma 2.6 and 2. that we can ensure the regularity of the convex potential v_1 we would produce after applying Lemma 2.6.

So long as $|\zeta_1(0)| \leq 1/27$, 1. is satisfied.

Now, let us move to understanding point 2. By (2.18),

$$\left\| u_1(\tilde{x}) - \frac{1}{2} |M\tilde{x}|^2 \right\|_{L^\infty(M^{-1}B_4 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \leq \frac{1}{2} \check{\eta},$$

from which, using our estimate on M, we find that

$$\left\| u_1(\tilde{x}) - \frac{1}{2} |\tilde{x}|^2 \right\|_{L^\infty(B_3 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \leq \frac{1}{2} \check{\eta} + C_5 \check{\delta} \leq \check{\eta}. \tag{2.22}$$

Applying Lemma 2.6 and arguing as before, (2.11) becomes

$$\left\| v_1(\tilde{x}) - \frac{1}{2} |\tilde{x}|^2 \right\|_{L^\infty(B_{1/6})} \leq C_0(\check{\eta} + \omega(\check{\delta}) + |\zeta_1(0)|). \tag{2.23}$$

Let $\rho \leq 1/27$ is the largest replacement for $|\zeta_1(0)|$ in (2.23) that permits (2.12) with v_1 replacing $\tilde{v}'$. In order to continue with Step 2, we need that

$$|\zeta_1(0)| \leq \rho.$$

Notice that if we decrease $\check{\eta}$ (and then necessarily $\check{\delta}$), we can increase ρ. In particular, we can ensure that

$$\rho \gg \check{\eta}^{1/2} + \check{\delta}. \tag{2.24}$$

So provided that $\check{\eta}, \check{\delta}$, and r are sufficiently small to guarantee 1., we can indeed continue and find A_1 and M_1 and define u_2, $\mathcal{C}_2$, $\mathcal{K}_2$, γ_2, ζ_2, f_2, and g_2. Notice that the only differences between this family and its predecessor is that $|\zeta_2(0)|$ will increase:

$$\frac{1}{C_4 h^{1/2}}|\zeta_1(0)| \leq |\zeta_2(0)| \leq \frac{C_4}{h^{1/2}}|\zeta_1(0)|$$

and the Hölder semi-norm of the lower boundaries of $\mathcal{C}_2$ and $\mathcal{K}_2$ will decrease:

$$[\nabla\gamma_2]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\zeta_2]_{C^{0,\alpha}(\mathcal{B}_3)} \leq c_0\delta_1 = c_0^2\check{\delta}.$$

Hence, if $|\zeta_2(0)| \leq \rho$, then we can repeat our procedure again and iterate further.

Suppose $\check{\zeta}(0) \neq 0$, i.e., $y^0 \notin \{y_n = \zeta(y_0)\}$. Then, there will be a first time $k \geq 2$ at which

$$|\zeta_k(0)| > \rho,$$

and we can no longer continue our iterative procedure. That said, recalling how we proved (2.22), we have that

$$\left\| u_k(\tilde{x}) - \frac{1}{2}|\tilde{x}|^2 \right\|_{L^\infty(B_3 \cap \{\tilde{x}_n \geq \gamma_k(\tilde{x}_0)\})} \leq \check{\eta}.$$

Consequently,

$$\partial u_k(\tilde{x}) \subset \left\{ \tilde{y}_n \geq -4\check{\delta} - \frac{5}{2}\check{\eta}^{1/2} \right\}$$

for all $\tilde{x} \in \{\tilde{x}_n \geq \gamma_k(\tilde{x}_0) + \check{\eta}^{1/2}\} \cap \mathcal{C}_k$ (cf. (2.6)). On the other hand,

$$|\mathcal{C}_k \setminus \{\tilde{x}_n \geq \gamma_k(\tilde{x}_0) + \check{\eta}^{1/2}\}| \leq C\check{\eta}^{1/2}$$

and, recalling (2.24),

$$\left| \mathcal{K}_k \setminus \left\{ y_n \geq -4\check{\delta} - \frac{5}{2}\check{\eta}^{1/2} \right\} \right| \geq c\left(\rho - 8\check{\delta} - \frac{5}{2}\check{\eta}^{1/2} \right) > 0.$$

But these two inequalities together violate the transport condition $(\nabla u_k)_\# f_k = g_k$ if $\check{\delta}$ and $\check{\eta}$ are sufficiently small (again, recall (2.24)).

As $\check{\zeta}(0) = 0$ (that is, $y^0 \in \{y_n = \zeta(y_0)\}$), we can iterate indefinitely. In turn, for all $k \geq 1$, we have determinant one matrices A_k and M_k such that

$$\frac{1}{C_4}\,\mathrm{Id} \leq A_k M_k \leq C_4\,\mathrm{Id}$$

and

$$D_k B_{h^{k/2}/7^{1/2}} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset S_{u_1}(h^k) \subset D_k B_{7^{1/2}h^{k/2}} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\}$$

where $D_k := A_1 M_1 \cdots A_{k-1} M_{k-1} A_k$. Hence,

$$B_{(h^{1/2}/3C_4)^k} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset S_{u_1}(h^k) \subset B_{(3C_4 h^{1/2})^k} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \quad \forall k \geq 1.$$

Thus, fixing $\beta \in (0,1)$ and then choosing h sufficiently small and $d := h^{1/2}/3C_4$, it follows that

$$\|u_1\|_{L^\infty(B_{d^k} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \leq d^{(1+\beta)k}.$$

Since x^0 was an arbitrary point on $B_r \cap \{x_n = \gamma(x_0)\}$, we have that $u \in C^{1,\beta}(B_r \cap \{x_n = \gamma(x_0)\})$.

– *Case 2:* $x^0 \in \{x_n > \gamma(x_0)\}$.

– *Step 4.2: A change of variables.*

Notice that in Case 1, we showed that

$$\partial u(B_r \cap \{x_n = \gamma(x_0)\}) = \nabla u(B_r \cap \{x_n = \gamma(x_0)\}) \subset B_{1/2} \cap \{y_n = \zeta(y_0)\}.$$

By duality, that is, considering inverse transport ∇u^*, it follows that

$$\partial u(B_r \cap \{x_n > \gamma(x_0)\}) \subset B_{1/2} \cap \{y_n > \zeta(y_0)\}. \tag{2.25}$$

Just as before, we find a determinant one matrix M such that the matrix $M^{-1}A^{-1}$ is symmetric and has eigenvectors $\{e_1, \dots, e_{n-1}, \mathrm{e}_n\}$ for which $\mathrm{e}_n \cdot e_i = 0$ for all $i = 1, \dots, n-1$. However, now that $\check{\gamma}(0) \neq 0$, we find that

$$|M - \mathrm{Id}\,| \leq C_3(|\check{\gamma}(0)| + \check{\delta}) < C_3(|\check{\gamma}(0)| + 2\check{\delta}) < C_3(\rho + \check{\delta}). \tag{2.26}$$

If we define γ_1 and ζ_1 as we did in Step 4.1, then

$$|\nabla\zeta_1(0)| \leq C_6|\nabla\check{\zeta}(0)|.$$

There are two subcases two consider: 1. $C_6 \leq 1$ and 2. $C_6 > 1$.

In Subcase 1, we consider the same change of variables as we did in Step 4.1. Then, from (2.20), (2.21), and construction,

$$B_{1/3} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset \mathcal{C}_1 \subset B_3 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\}$$

and

$$B_{1/3} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\} \subset \mathcal{K}_1 \subset B_3 \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\}$$

provided the right-hand side of (2.26) is sufficiently small. Additionally,

$$\frac{1}{C_7 h^{1/2}}|\check{\gamma}(0)| \leq |\gamma_1(0)| \leq \frac{C_7}{h^{1/2}}|\check{\gamma}(0)|, \quad \frac{1}{C_7 h^{1/2}}|\check{\zeta}(0)| \leq |\zeta_1(0)| \leq \frac{C_7}{h^{1/2}}|\check{\zeta}(0)|,$$
$$\nabla\gamma_1(0) = 0, \quad |\nabla\zeta_1(0)| \leq \check{\delta},$$

and

$$[\nabla\gamma_1]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\zeta_1]_{C^{0,\alpha}(\mathcal{B}_3)} \leq c_1\check{\delta} \tag{2.27}$$

for $c_1 \ll 1$, taking h smaller.

In Subcase 2, we additionally apply a shearing transformation L^* to swap which side has a horizontal tangent at 0 for the function defining the lower boundary. More precisely, there exists a shearing transformation L so that

$$\nabla\zeta_1(0) = 0 \quad \text{and} \quad |L - \mathrm{Id}\,| \leq C_6\check{\delta},$$

defining

$$\tilde{y} := \frac{1}{h^{1/2}} L^* M^* A \check{y}$$

and letting $\zeta_1 : \mathcal{B}_3 \to \mathbb{R}$ be such that

$$\{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\} = \frac{1}{h^{1/2}} L^* M^* A \{\check{y}_n \geq \check{\zeta}(\check{y}_0)\}.$$

So considering the change of variables

$$\tilde{x} := \frac{1}{h^{1/2}} L^{-1} M^{-1} A^{-1} \check{x} \quad \text{and} \quad \tilde{y} := \frac{1}{h^{1/2}} L^* M^* A \check{y},$$

we define

$$u_1(\tilde{x}) := \frac{1}{h} \check{u}(h^{1/2} AML\tilde{x})$$

and set

$$\mathcal{C}_1 := S_{u_1}(1) \quad \text{and} \quad \mathcal{K}_1 := \partial_* u_1(S_{u_1}(1)).$$

As $\det(L) = \det(M) = \det(A) = 1$, we deduce that

$$(\nabla u_1)_\# f_1 = g_1$$

with

$$f_1(\tilde{x}) := \check{f}(h^{1/2} AML\tilde{x}) \mathbf{1}_{\mathcal{C}_1} \quad \text{and} \quad g_1(\tilde{y}) := \check{g}(h^{1/2} A^{-1} (M^*)^{-1} (L^*)^{-1} \tilde{y}) \mathbf{1}_{\mathcal{K}_1}.$$

From (2.20), (2.21), and our estimates on M and L,

$$B_{1/3} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset \mathcal{C}_1 \subset B_3 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\}$$

and

$$B_{1/3} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\} \subset \mathcal{K}_1 \subset B_3 \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\}.$$

Additionally,

$$\frac{1}{C_8 h^{1/2}} |\check{\gamma}(0)| \leq |\gamma_1(0)| \leq \frac{C_8}{h^{1/2}} |\check{\gamma}(0)|, \quad \frac{1}{C_8 h^{1/2}} |\check{\zeta}(0)| \leq |\zeta_1(0)| \leq \frac{C_8}{h^{1/2}} |\check{\zeta}(0)|,$$

$$|\nabla \gamma_1(0)| \leq \check{\delta}, \quad \nabla \zeta_1(0) = 0,$$

and

$$[\nabla \gamma_1]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla \zeta_1]_{C^{0,\alpha}(\mathcal{B}_3)} \leq c_1 \check{\delta}$$

where for the inequality $|\nabla \gamma_1(0)| \leq \check{\delta}$, we have used (2.27).

– *Step 5.2: An iteration scheme.*

In order run Steps 2 through 4.2 on u_1, $\mathcal{C}_1$, $\mathcal{K}_1$, γ_1, ζ_1, f_1, and g_1, like before, we need to ensure two things: 1. that the hypotheses of Lemma 2.6 and 2. that we can ensure the regularity of the convex potential v_1 we would produce after applying Lemma 2.6.

So long as $|\gamma_1(0)|, |\zeta_1(0)| \leq 1/27$, 1. is satisfied.

Now let us move to understanding point 2. By (2.18),

$$\left\| u_1(\tilde{x}) - \frac{1}{2} |ML\tilde{x}|^2 \right\|_{L^\infty(L^{-1}M^{-1}B_4 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \leq \frac{1}{2} \check{\eta},$$

from which, using our estimates on M and L, we find that

$$\left\| u_1(\tilde{x}) - \frac{1}{2} |\tilde{x}|^2 \right\|_{L^\infty(B_3 \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \leq \check{\eta} + C_9 |\check{\gamma}(0)|.$$

Applying Lemma 2.6 and arguing as before, (2.11) becomes

$$\begin{aligned}\left\| v_1(\tilde{x}) - \frac{1}{2}|\tilde{x}|^2 \right\|_{L^\infty(B_{1/6})} &\leq C_0(\check{\eta} + \omega(\check{\delta}) + C_9|\check{\gamma}(0)| + |\gamma_1(0)| + |\zeta_1(0)| + \check{\delta}) \\ &\leq \tilde{C}_0(\check{\eta} + \omega(\check{\delta}) + |\check{\gamma}(0)| + |\gamma_1(0)| + |\zeta_1(0)|).\end{aligned}$$

In this case, we need

$$|\check{\gamma}(0)| + |\gamma_1(0)| + |\zeta_1(0)| \leq \rho$$

to proceed, decreasing ρ to account for the larger factor $\tilde{C}_0$. Recall that

$$\rho \gg \check{\eta}^{1/2} + \check{\delta}. \tag{2.28}$$

Hence, if $\check{\eta}, \check{\delta}$, and r are sufficiently small, we can indeed continue and find A_1 a symmetric, determinant one matrix such that

$$\frac{1}{C_1^{1/2}} \operatorname{Id} \leq A_1 \leq C_1^{1/2} \operatorname{Id},$$

$$A_1 B_{(h/7)^{1/2}} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\} \subset S_{u_1}(h) \subset A_1 B_{(7h)^{1/2}} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\},$$

$$A_1^{-1} B_{(h/7)^{1/2}} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_1)\} \subset \partial_* u_1(S_{u_1}(h)) \subset A_1^{-1} B_{(7h)^{1/2}} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\},$$

and

$$\begin{aligned}&\left\| u_1(\tilde{x}) - \frac{1}{2}|A_1^{-1}\tilde{x}|^2 \right\|_{L^\infty(A_1 B_{(16h)^{1/2}} \cap \{\tilde{x}_n \geq \gamma_1(\tilde{x}_0)\})} \\ &\qquad + \left\| u_1^*(\tilde{y}) - \frac{1}{2}|A_1\tilde{y}|^2 \right\|_{L^\infty(A_1^{-1} B_{(16h)^{1/2}} \cap \{\tilde{y}_n \geq \zeta_1(\tilde{y}_0)\})} \leq \check{\eta} h.\end{aligned}$$

Using the same construction as before, we build M_1 and if needed L_1 (this time however L_1 will make the tangent plane at 0 to lower boundary of the source horizontal and the tangent plane at 0 to the lower boundary of the target smaller than $\check{\delta}$) and define u_2, $\mathcal{C}_2$, $\mathcal{K}_2$, γ_2, ζ_2, f_2, and g_2. Now

$$|M_1 - \operatorname{Id}| \leq C_3(|\gamma_1(0)| + 2\check{\delta}) < C_3(\rho + \check{\delta}),$$

$$[\nabla\gamma_2]_{C^{0,\alpha}(\mathcal{B}_3)} + [\nabla\zeta_2]_{C^{0,\alpha}(\mathcal{B}_3)} \leq c_1^2\check{\delta},$$

$$\frac{1}{C_8 h^{1/2}}|\gamma_1(0)| \leq |\gamma_2(0)| \leq \frac{C_8}{h^{1/2}}|\gamma_1(0)|, \quad \text{and} \quad \frac{1}{C_8 h^{1/2}}|\zeta_1(0)| \leq |\zeta_2(0)| \leq \frac{C_8}{h^{1/2}}|\zeta_1(0)|.$$

Continuing, there will be a first time $k \geq 2$ when

$$2(|\zeta_k(0)| + |\gamma_k(0)|) \geq |\gamma_{k-1}(0)| + |\gamma_k(0)| + |\zeta_k(0)| > \rho.$$

At this point, we go back to u_{k-1} and consider

$$\tilde{u}(\tilde{x}) := \frac{1}{h} u_{k-1}(h^{1/2} A_{k-1}\tilde{x})$$

(and, correspondingly, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{K}}$, $\tilde{\gamma}$, and $\tilde{\zeta}$) rather than u_k, forgetting about M_{k-1} and L_{k-1}. Notice that

$$B_{1/3} \cap \{\tilde{x}_n \geq \tilde{\gamma}(\tilde{x}_0)\} \subset \mathcal{C}_1 \subset B_3 \cap \{\tilde{x}_n \geq \tilde{\gamma}(\tilde{x}_0)\},$$

$$B_{1/3} \cap \{\tilde{y}_n \geq \tilde{\zeta}(\tilde{y}_0)\} \subset \mathcal{K}_1 \subset B_3 \cap \{\tilde{y}_n \geq \tilde{\zeta}(\tilde{y}_0)\},$$

and

$$\left\| \tilde{u}(\tilde{x}) - \frac{1}{2}|\tilde{x}|^2 \right\|_{L^\infty(B_3 \cap \{\tilde{x}_n \geq \tilde{\gamma}(\tilde{x}_0)\})} + \left\| \tilde{u}^*(\tilde{y}) - \frac{1}{2}|\tilde{y}|^2 \right\|_{L^\infty(B_3 \cap \{\tilde{y}_n \geq \tilde{\zeta}(\tilde{y}_0)\})} \leq \check{\eta}. \tag{2.29}$$

Moreover,

$$C(|\tilde{\gamma}(0)| + |\tilde{\zeta}(0)|) \geq |\gamma_k(0)| + |\zeta_k(0)|.$$

Hence, using (2.29), arguing as we did to prove (2.7), (2.25), and recalling that (2.28), we deduce that

$$B_{c\rho} \subset \tilde{\mathcal{C}} \quad \text{and} \quad B_{c\rho} \subset \tilde{\mathcal{K}}.$$

So we are in an interior situation, and taking $\check{\eta}$ (and also $\check{\delta}$) sufficiently small depending on ρ, we can apply the arguments of [7, Theorem 4.3] to conclude that $u \in C^{1,\beta}(x^0)$. As $x^0 \in B_r \cap \{x_n > \gamma(x_0)\}$ was arbitrary, the theorem holds. □

3. Perturbations in Non-regular Domains

In the previous section, we considered perturbations in regions of domains that are at least $C^{1,\alpha}$, and in the case of Theorem 2.5, we additionally considered non-constant densities. The next natural question is, what can be said about domain and density perturbations of less regular portions of a domain? We have seen, in some sense, that corners destroy regularity. So this question is unlikely to have a satisfactory answer. Yet if we consider the simple situation of rectangles in $\mathbb{R}^2$, the highly symmetric nature of these domains might be leveraged to say something of interest.

First, we remark that Theorem 2.5 can be extended to domains in two dimensions that are deformation of domains with 90 degree corners. The fundamental domain in Theorem 2.5 is an upper half ball; in the sense given in its hypotheses, $\mathcal{C}$ and $\mathcal{K}$ are comparable to upper half balls. If $\gamma, \zeta \equiv 0$, then the points on $\{x_2 = \gamma(x_1)\}$ are interior points for the optimal transport for the data reflected over horizontal lines, and regularity follows from [8, Proposition 2]. In Theorem 2.5, the regularity of u at/near the lower boundary is inherited from the interior regularity of a potential to an approximating problem that takes advantage of this "reflection symmetry yields regularity" argument. The same strategy will extend Theorem 2.5 when considering an upper quarter ball as a fundamental domain. Just as points on flat boundaries can be turned into interior points, 90 degree corners and points near these corners can be turn into interior points. (See, e.g., Lemma 3.1.) We leave the details of Theorem 2.5's extension to the interested reader.

Now let us move to considering higher order density perturbations in corners: given two densities $f, g \in C^\infty(\overline{Q})$ bounded away from zero and satisfying the mass balance condition $\|f\|_{L^1(Q)} = \|g\|_{L^1(Q)}$ where $Q := (0,1) \times (0,1)$, is the optimal transport T taking f to g of class $C^\infty(\overline{Q})$?

Set

$$\Upsilon_{\mathrm{b}} := (0,1) \times \{0\}, \quad \Upsilon_{\mathrm{t}} := (0,1) \times \{1\}, \quad \Upsilon_{\mathrm{l}} := \{0\} \times (0,1), \quad \text{and} \quad \Upsilon_{\mathrm{r}} := \{1\} \times (0,1).$$

In what follows, we let C be a generic positive constant; it may change from line to line, and its dependences, if any, will either be clear from context or explicitly given.

First, notice that $T \in C^\infty_{\mathrm{loc}}(Q) \cap C^{0,\sigma}(\overline{Q})$ for some $\sigma < 1$ (see [2, 3]). So to start, we show, taking advantage of the reflection symmetries of Q, that the non-uniform convexity of Q does not prohibit $C^{2,\alpha}$-regularity given α-Hölder continuous densities on $\overline{Q}$ bounded away from zero. In other words, the symmetries of Q allow us to recover the same regularity up to the boundary of our transport as we would had our source and target domains been uniformly convex. (See [4].)

Lemma 3.1. *Let $f, g \in C^{0,\alpha}(\overline{Q})$ be bounded away from zero and satisfy the mass balance condition $\|f\|_{L^1(Q)} = \|g\|_{L^1(Q)}$. The optimal transport T is a diffeomorphism of class $C^{1,\alpha}(\overline{Q})$. Moreover, T maps each segment of the boundary of Q diffeomorphically to itself.*

Proof. First, by [3], $T : \overline{Q} \to \overline{Q}$ is a bi-Hölder continuous homeomorphism. Now set

$$Q'' := (-1, 1) \times (-1, 1),$$

and let f'' be the even reflection of f around the origin to Q'':

$$f''(x) := \begin{cases} f(x_1, x_2) & \text{in } [0,1] \times [0,1] \\ f(x_1, -x_2) & \text{in } [0,1] \times (0,-1] \\ f(-x_1, x_2) & \text{in } [-1,0) \times [0,1] \\ f(-x_1, -x_2) & \text{in } [-1,0) \times (0,-1]. \end{cases}$$

Also, let g'' be the even reflection of g around the origin to Q''. By construction, f'' and g'' are of class $C^{0,\alpha}(Q'')$; and so, $T'' \in C^{1,\alpha}(B_{3/4})$, by [2], where T'' is the optimal transport taking f'' to g''. By symmetry, $T''([-1,1] \times \{0\}) = [-1,1] \times \{0\}$, $T''(\{0\} \times [-1,1]) = \{0\} \times [-1,1]$, and the restriction of T'' to $\overline{Q}$ is T, the optimal transport taking f to g. It follows that $T \in C^{1,\alpha}(B_{3/4} \cap Q)$. The lemma then follows after similarly reflecting f and g evenly around the points $(1,0), (0,1)$, and $(1,1)$. □

Remark 3.2. We can also see that $T = \nabla u$ maps each segment of the boundary of Q to itself and fixes the corners of Q via a local argument. If, say, $\nabla u(x) \in \Upsilon_{\mathrm{r}}$ for some $x \in \Upsilon_{\mathrm{b}}$, then $\nabla u(\ell_x) \subset \overline{\Upsilon}_{\mathrm{r}}$ for some ℓ_x non-empty subsegment of Υ_{b} containing x (possibly as an endpoint). Thus, $\partial_1 u(\cdot, 0)$ is constant on ℓ_x, or, equivalently, $u|_{\ell_x}$ is linear. However, this contradicts the strict convexity of u along the boundary of Q given by [3]. Finally, $\nabla u(x) \notin \Upsilon_{\mathrm{t}}$ for any $x \in \Upsilon_{\mathrm{b}}$ by the monotonicity of ∇u.

As we are working in two dimensions, rather than consider a Monge-Ampère equation, we can instead consider a quasi-linear, uniformly elliptic equation for the partial Legendre transform $u^\star$ of u; and after absorbing the coefficients' dependences on $u^\star$ at the expense of their regularity, we can consider a linear, uniformly elliptic equation. This observation will play a key role in answering our question.

Theorem 3.3. *Let $f, g \in C^{1,\alpha}(\overline{Q})$ be bounded away from zero and satisfy the mass balance condition $\|f\|_{L^1(Q)} = \|g\|_{L^1(Q)}$. The optimal transport T is a diffeomorphism of class $C^{2,\alpha}(\overline{Q})$. Moreover, T maps each segment of the boundary of Q diffeomorphically to itself.*

Proof. By Lemma 3.1, given any f and g of class $C^{1,\alpha}(\overline{Q})$ bounded away from zero, any convex potential u of the optimal transport T taking f to g is $C^{2,\alpha}(\overline{Q})$. Moreover, because T maps each segment of the boundary of Q to itself, we see that $\partial_\nu u = 0$ on $\Upsilon_{\mathrm{b}} \cup \Upsilon_{\mathrm{l}}$ and $\partial_\nu u = 1$ on $\Upsilon_{\mathrm{t}} \cup \Upsilon_{\mathrm{r}}$. In particular, $\nabla u(0) = 0$. Hence,

$$\begin{cases} \det D^2 u = f/g(\nabla u) & \text{in } Q \\ \partial_\nu u = 0 & \text{on } \Upsilon_{\mathrm{b}} \cup \Upsilon_{\mathrm{l}} \\ \partial_\nu u = 1 & \text{on } \Upsilon_{\mathrm{t}} \cup \Upsilon_{\mathrm{r}}. \end{cases} \tag{3.1}$$

Furthermore, u is uniformly convex as $f/g > 0$. In other words,

$$0 < \frac{1}{C} \leq \partial_{ii} u(x) \leq C \quad \forall x \in \overline{Q}. \tag{3.2}$$

Now let $u^\star$ be the partial Legendre transform of u in the e_1-direction:

$$u^\star(p, x_2) := \sup_{x_1 \in \overline{Q}_{x_2}} \{px_1 - u(x_1, x_2)\}.$$

Here, $\overline{Q}_{x_2}$ is the horizontal slice of $\overline{Q}$ at height x_2. Notice that the point $x_1 = X_1(p, x_2)$ where this supremum is attained is characterized by the equation

$$\partial_1 u(X_1(p, x_2), x_2) = p. \tag{3.3}$$

Since $u(\cdot, x_2)$ is strictly convex and of class $C^1(\overline{Q}_{x_2})$, we have that $\partial_1 u(\cdot, x_2)$ is injective and (3.3) is uniquely solvable given a pair $(p, x_2) \in \overline{Q}$. The map $(x_1, x_2) \mapsto (\partial_1 u(x_1, x_2), x_2)$ takes $\overline{Q}$ to $\overline{Q}$ as $T = \nabla u$ maps $\overline{Q}$ to $\overline{Q}$. Recall that the first partial derivatives of $u^\star$ are related to the first partial derivatives of u by the equations

$$\partial_1 u^\star(p, x_2) = X_1(p, x_2) \quad \text{and} \quad \partial_2 u^\star(p, x_2) = -\partial_2 u(X_1(p, x_2), x_2),$$

while the pure second partial derivatives of $u^\star$ are related to the pure second partial derivatives of u by the equations

$$\partial_{11} u^\star(p, x_2) = \frac{1}{\partial_{11} u(X_1(p, x_2), x_2)} \quad \text{and} \quad \partial_{22} u^\star(p, y) = \left[\frac{(\partial_{12} u)^2}{\partial_{11} u} - \partial_{22} u\right](X_1(p, x_2), x_2).$$

Therefore, using (3.1), (3.2), and (3.3), it follows that

$$\begin{cases} f(\partial_1 u^\star, x_2)\partial_{11} u^\star + g(p, -\partial_2 u^\star)\partial_{22} u^\star = 0 & \text{in } Q \\ \partial_\nu u^\star = 0 & \text{on } \Upsilon_\mathrm{b} \cup \Upsilon_\mathrm{l} \\ \partial_\nu u^\star = 1 & \text{on } \Upsilon_\mathrm{t} \cup \Upsilon_\mathrm{r}. \end{cases} \tag{3.4}$$

Applying Proposition 3.4 to $v = u^\star$, $a_1 = a_1(p, x_2) = f(\partial_1 u^\star(p, x_2), x_2)$, and $a_2 = a_2(p, x_2) = g(p, -\partial_2 u^\star(p, x_2))$, we see that $u^\star$ is of class $C^{3,\alpha}(\overline{Q}_{3/4})$. By symmetry, we can treat each corner as the origin; it follows that $\|u^\star\|_{C^{3,\alpha}(\overline{Q})} \leq C$.

Finally, as u is uniformly convex ((3.2)), u and $u^\star$ have the same regularity on the closure of Q. □

Let $Q_r := (0, r) \times (0, r) = rQ$.

Proposition 3.4. *Let $v \in C(\overline{Q})$ be such that $\|v\|_{L^\infty(\overline{Q})} \leq 1$ and*

$$\begin{cases} \operatorname{tr}(AD^2 v) = 0 & \text{in } Q \\ \partial_\nu v = 0 & \text{on } \Upsilon_\mathrm{b} \cup \Upsilon_\mathrm{l} \end{cases}$$

where

$$\lambda \operatorname{Id} \leq A := \operatorname{diag}(a_1, a_2) \leq \Lambda \operatorname{Id}.$$

If $\|A\|_{C^{1,\alpha}(\overline{Q})} \leq \Lambda$, then

$$\|v\|_{C^{3,\alpha}(\overline{Q}_{3/4})} \leq C$$

for some constant $C = C(\lambda, \Lambda, \alpha) > 0$.

Proof. Up to a diagonal transformation, we can assume that $a_i(0) = 1$ for $i = 1, 2$. Then, up to zeroth order scaling, i.e., considering $v_r(x) := v(rx)$ and $A_r(x) := A(rx)$, we can assume that

$$\|A - \operatorname{Id}\|_{C^{1,\alpha}(\overline{Q})} \leq \varepsilon$$

for some $\varepsilon > 0$ that will be chosen.

Let P be a degree three polynomial such that

$$\partial_1 P(0,\cdot) = \partial_2 P(\cdot,0) \equiv 0.$$

Then, P takes the form

$$P(x) = p_0 + p_{2,1}x_1^2 + p_{2,2}x_2^2 + p_{3,1}x_1^3 + p_{3,2}x_2^3.$$

Set

$$\|P\| := \max_{m,j} |p_{m,j}|.$$

Taylor expanding A around the origin, we see that

$$\operatorname{tr}(AD^2P) = \hat{P} + h$$

where $\hat{P}$ is a degree one polynomial with coefficients

$$\begin{cases} d_0 &= 2p_{2,1} + 2p_{2,2} \\ d_{1,1} &= 6p_{3,1} + 2p_{2,1}\partial_1 a_1(0) + 2p_{2,2}\partial_1 a_2(0) \\ d_{1,2} &= 6p_{3,2} + 2p_{2,1}\partial_2 a_1(0) + 2p_{2,2}\partial_2 a_2(0) \end{cases} \tag{3.5}$$

and h is such that

$$|h(x)| \leq C\varepsilon|x|^{1+\alpha} \tag{3.6}$$

with $C = C(\|P\|) > 0$. Notice that given any triplet $(d_0, d_{1,1}, d_{1,2})$, the system (3.5) is uniquely solvable after choosing $p_{2,1}$. Indeed,

$$\begin{cases} 2p_{2,2} = d_0 - 2p_{2,1} \\ 6p_{3,1} = d_{1,1} - (d_0 - 2p_{2,1})\partial_1 a_2(0) - 2p_{2,1}\partial_1 a_1(0) \\ 6p_{3,2} = d_{1,2} - (d_0 - 2p_{2,1})\partial_2 a_2(0) - 2p_{2,1}\partial_2 a_1(0). \end{cases}$$

Definition 3.5. We call a degree three polynomial P approximating for v at zero if $\partial_1 P(0,\cdot) = \partial_2 P(\cdot,0) \equiv 0$ and $\hat{P} \equiv 0$.

Recall $Q'' = (-1,1) \times (-1,1)$, and let $Q_r'' := rQ''$.

Lemma 3.6. *Assume that for some $r \leq 1$ and some approximating polynomial P for v at zero with $\|P\| \leq 1$, we have that*

$$\|v - P\|_{L^\infty(\overline{Q}_r)} \leq r^{3+\alpha}.$$

Then, there exists an approximating polynomial $\bar{P}$ for v at zero such that

$$\|v - \bar{P}\|_{L^\infty(\overline{Q}_{\rho r})} \leq (\rho r)^{3+\alpha}$$

and

$$\|P - \bar{P}\|_{L^\infty(Q_r'')} \leq Cr^{3+\alpha}$$

for some constants $\rho = \rho(\lambda,\Lambda,\alpha), C = C(\lambda,\Lambda,\alpha) > 0$.

Proof. Let

$$\tilde{v}(x) := \frac{[v-P](rx)}{r^{3+\alpha}} \quad \text{and} \quad \tilde{A}(x) := A(rx).$$

Then,

$$\|\tilde{v}\|_{L^\infty(\overline{Q})} \leq 1.$$

Since P is an approximating polynomial for v at zero, we have that

$$\begin{cases} \operatorname{tr}(\tilde{A}D^2\tilde{v}) = \tilde{h} & \text{in } Q \\ \partial_\nu \tilde{v} = 0 & \text{on } \Upsilon_{\mathrm{b}} \cup \Upsilon_{\mathrm{l}}, \end{cases}$$

and from (3.6),

$$|\tilde{h}| \leq C\varepsilon.$$

Now consider the even reflections of $\tilde{A}, \tilde{v}$, and $\tilde{h}$ around the origin, which we denote by $\tilde{A}'', \tilde{v}''$, and $\tilde{h}''$. It follows that (see, e.g., [11, Proposition 4.1]),

$$\operatorname{tr}(\tilde{A}''D^2\tilde{v}'') = \tilde{h}'' \text{ in } Q''.$$

Since $\tilde{A}''$ are uniformly Hölder continuous and $\tilde{v}''$ and $\tilde{h}''$ are uniformly bounded in Q'', we deduce that $\tilde{v}''$ are locally uniformly Hölder continuous in Q''. (Recall all these functions depend on our choice of ε.) By compactness, as ε converges to zero, we then find that, up to subsequences, $\tilde{v}''$ must converge uniformly in Q''_ρ for every $\rho < 1$ to a function v_0 that is harmonic in Q'' and bounded by 1. Thus, since $\tilde{v}''|_{Q_\rho} = \tilde{v}$,

$$\|\tilde{v} - P_0\|_{L^\infty(\overline{Q}_\rho)} \leq \|\tilde{v} - v_0\|_{L^\infty(\overline{Q}_\rho)} + \|v_0 - P_0\|_{L^\infty(\overline{Q}_\rho)} \leq C\varepsilon + C\rho^4 \leq \frac{2}{3}\rho^{3+\alpha} \tag{3.7}$$

if $\varepsilon, \rho > 0$ are chosen sufficiently small. Here, P_0 is the harmonic degree three Taylor polynomial of v_0 at the origin. Furthermore, $\partial_1 P_0(0, \cdot) = \partial_2 P_0(\cdot, 0) \equiv 0$ by symmetry. Hence, P_0 has no linear or cubic part, no mixed two degree part, and the remaining two (pure) second degree coefficients of P_0 are such that

$$0 = 2(p_0)_{2,1} + 2(p_0)_{2,2} \tag{3.8}$$

Rescaling, we determine that

$$\|v - P - r^{3+\alpha}P_0(\cdot/r)\|_{L^\infty(\overline{Q}_{\rho r})} \leq \frac{2}{3}(\rho r)^{3+\alpha}.$$

Unfortunately, the polynomial

$$P(x) + r^{3+\alpha}P_0(x/r)$$

is not necessarily approximating for v at zero. To make it approximating, we want to replace P_0 with a polynomial $\bar{P}_0$ whose coefficients satisfy

$$\begin{cases} 0 = 2(\bar{p}_0)_{2,1} + 2(\bar{p}_0)_{2,2} \\ 0 = 6(\bar{p}_0)_{3,1} + r2(\bar{p}_0)_{2,1}\partial_1 a_1(0) + r2(\bar{p}_0)_{2,2}\partial_1 a_2(0) \\ 0 = 6(\bar{p}_0)_{3,2} + r2(\bar{p}_0)_{2,1}\partial_2 a_1(0) + r2(\bar{p}_0)_{2,2}\partial_2 a_2(0). \end{cases} \tag{3.9}$$

Subtracting (3.9) and (3.8), we see that the coefficients for $P_0 - \bar{P}_0$ solve the system (3.9) with left-hand side

$$\begin{cases} d_0 &= 0 \\ d_{1,1} &= r2(p_0)_{2,1}\partial_1 a_1(0) + r2(p_0)_{2,2}\partial_1 a_2(0) \\ d_{1,2} &= r2(p_0)_{2,1}\partial_2 a_1(0) + r2(p_0)_{2,2}\partial_2 a_2(0). \end{cases}$$

Thus, after choosing $(\bar{p}_0)_{2,1} = (p_0)_{2,1}$, since $\max_{m,j} |d_{m,j}| \leq Cr\varepsilon$, it follows that $\bar{P}_0$ can be found so that

$$\|P_0 - \bar{P}_0\|_{L^\infty(Q'')} \leq C\varepsilon.$$

Finally, replacing P_0 with $\bar{P}_0$ in (3.7), we obtain the desired conclusion. Also, observe that

$$\bar{P}(x) := P(x) + r^{3+\alpha}\bar{P}_0(x/r)$$

is such that

$$\|\bar{P} - P\|_{L^\infty(Q''_r)} \leq Cr^{3+\alpha},$$

as desired. □

After multiplying v be a small constant, the hypotheses of Lemma 3.6 are satisfied with $P \equiv 0$ and $r = r_0$. Provided r_0 is sufficiently small (depending only on λ, Λ, and α), we can iteratively apply Lemma 3.6 with $r = r_0\rho^k$ to determine the existence of a limiting approximating polynomial P^0 for v at zero such that

$$\|P^0\| \leq C \quad \text{and} \quad \|v - P^0\|_{L^\infty(\overline{Q}_r)} \leq Cr^{3+\alpha} \quad \forall r \leq r_0.$$

Repeating a similar procedure at every point in $x \in \overline{Q}_{3/4}$, we find approximating polynomials P^x such that the above inequalities holds with the same constant $C > 0$, which implies that $v \in C^{3,\alpha}(\overline{Q}_{3/4})$, as desired. □

At this point, we might hope to prove higher order Schauder estimates and then bootstrap to show that $u \in C^\infty(\overline{Q})$. Recalling (3.4), the regularity of the coefficients of our equation is limited by the regularity of $\nabla u^\star$ up the boundary of Q. However, this strategy falls short at the next stage. The system of equations governing the existence of an approximating polynomial is degenerate; the normal derivative condition is too restrictive. A simple manifestation of this is seen by considering

$$\Delta v = x_1x_2 \text{ in } Q \quad \text{and} \quad \partial_1 v(0, x_2) = \partial_2 v(x_1, 0) \equiv 0.$$

A solution to this equation cannot be $C^4(\overline{Q})$. Indeed, if it were, taking ∂_{12}, we see that

$$\partial_{1112}v(0) + \partial_{2221}v(0) = 1.$$

Yet from the boundary data, we have that

$$\partial_{1112}v(0) + \partial_{2221}v(0) = 0.$$

This is impossible. An adaptation of this example shows that optimal transport maps from Q to itself may not be $C^3(\overline{Q})$ for generic (smooth) densities.

Theorem 3.7. *There exist $f, g : \overline{Q} \to \mathbb{R}$ two smooth densities bounded away from zero and satisfying the mass balance condition $\|f\|_{L^1(Q)} = \|g\|_{L^1(Q)}$ such that the optimal transport T taking f to g is of class $C^{2,\alpha}(\overline{Q})$ for every $\alpha < 1$ but not $C^3(\overline{Q})$.*

Proof. Let $f(x) := 1 + x_1x_2$ and $g(y) := 5/4$. By Theorem 3.3, a convex potential u defining T is of class $C^{3,\alpha}(\overline{Q})$ for all $\alpha < 1$. Now suppose, to the contrary, that $u \in C^4(\overline{Q})$. Then, $u^\star \in C^4(\overline{Q})$, and the first equation in (3.4) becomes

$$\partial_{11}u^\star + \frac{5}{4}\partial_{22}u^\star = -x_2\partial_1 u^\star \partial_{11}u^\star.$$

So differentiating in the e_1-direction and then in the e_2-direction, we see that

$$\partial_{1112}u^\star + \frac{5}{4}\partial_{2221}u^\star = -((\partial_{11}u^\star)^2 + 2x_2\partial_{11}u^\star\partial_{112}u^\star + \partial_1 u^\star\partial_{111}u^\star$$
$$+ x_2\partial_{12}u^\star\partial_{111}u^\star + x_2\partial_1 u^\star\partial_{1112}u^\star).$$

By the boundary conditions in (3.4) and recalling that $\partial_1 u^\star(0) = 0$, we deduce that

$$0 = \partial_{11} u^\star(0).$$

Yet this is impossible: by (3.2),

$$\partial_{11} u^\star(0) = \frac{1}{\partial_{11} u(0)} \geq \frac{1}{C} > 0.$$

□

Remark 3.8. Replacing f and g with $f_\varepsilon(x) := 1 + \varepsilon x_1 x_2$ and $g_\varepsilon(y) := 1 + \varepsilon/4$ yields the same contradiction as above, but with densities arbitrarily close (in C^∞) to 1.

Remark 3.9. We remark that attempting to run the same argument with u, instead of $u^\star$, does not in any obvious way lead to an analogous contradiction. In particular, taking ∂_{12} of $\det(D^2 u) = 4(1 + x_1 x_2)/5$ and arguing as in Theorem 3.7, we obtain the equation $\partial_{111} u(0) \partial_{222} u(0) = 4/5$. But there is no apparent reason this is false.

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